Word Problems: Acquiring Algebraic Thinking

John Mighton and Barry Garelick

researchED/Toronto

June 7, 2025

Thank you for attending our session, and also I want to thank researchED for the opportunity to present at this conference. My name is Barry Garelick; I'm happy to be in Toronto; the last time I was here was in 1965. I knew I would be back someday and this is it.

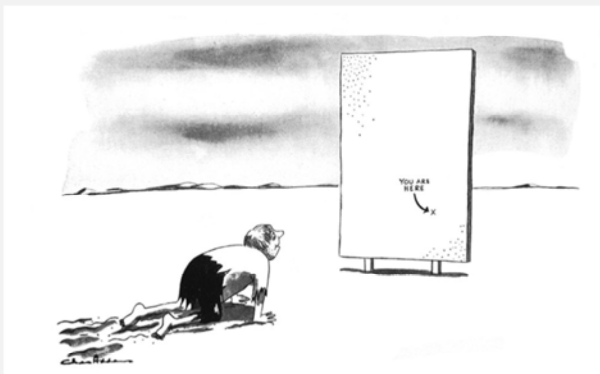
So with that, let me first introduce someone you all know: John Mighton. John is an award-winning mathematician, playwright and best-selling author, who founded JUMP Math as a charity in 2002. He is internationally recognized for his groundbreaking work building children's confidence, skills, and success in math.

As for me, I majored in mathematics at University of Michigan and used mathematics in the field of environmental protection. I became involved in math education advocacy when I saw how my daughter was being taught math. That inspired me to teach math when I retired: 7th and 8th grade math and algebra.

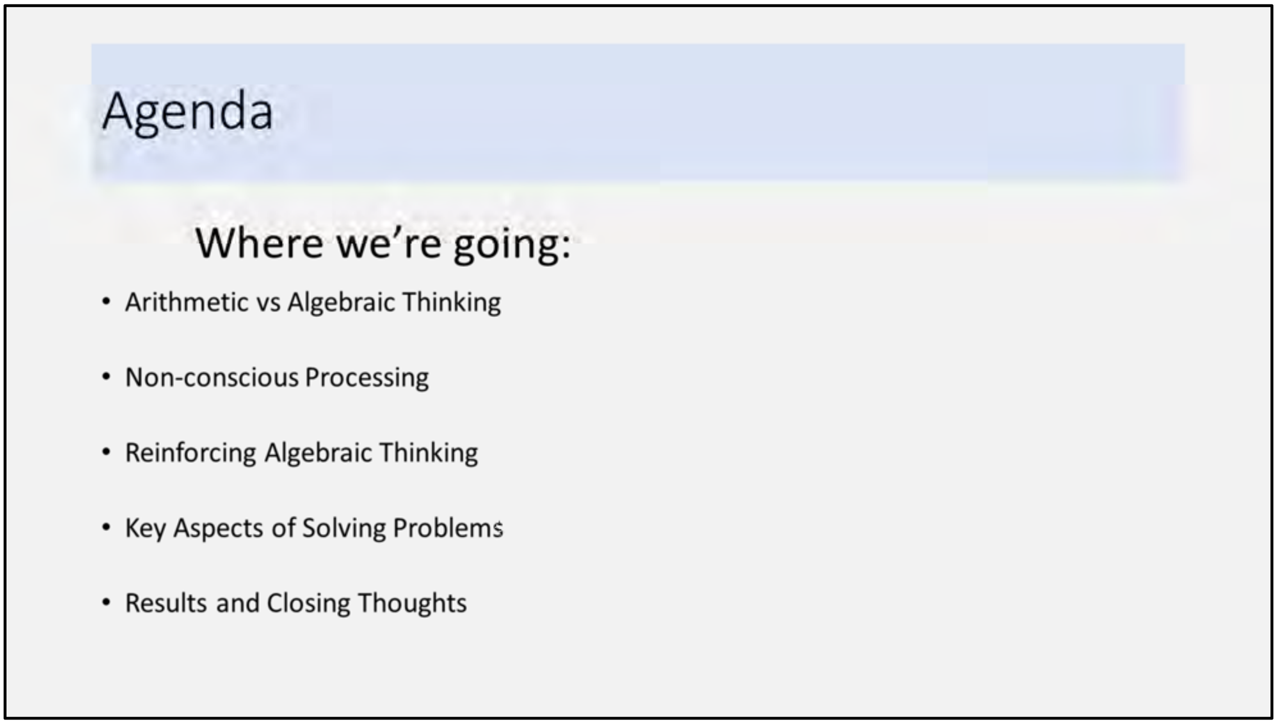
I retired for the second time in my life a few years ago but I've continued my work in math education. In fact, I've had the great honor of working with John in developing JUMP Math lessons on word problems, for what will be JUMP's upcoming algebra course. In today's presentation John and I will provide some of the details of what has guided our thinking in designing the course.

Agenda

Where we are:



Here’s where we are



And here's where we're going:

We'll start with the difference between arithmetic and algebraic thinking,

Then, how we make non-conscious choices of what methods to use in solving problems.

John will describe how we increase students' algebraic repertoire through scaffolding and variation theory

We'll take a closer look at key aspects of solving problems that guide our teaching, including "imitation", "desirable difficulties" and "staying in the zone".

Finally, we'll discuss some results of JUMP Math's programs, and closing thoughts. We'll leave time at the end to answer any questions you may have. So with that, let's get started

From arithmetic to algebraic thinking

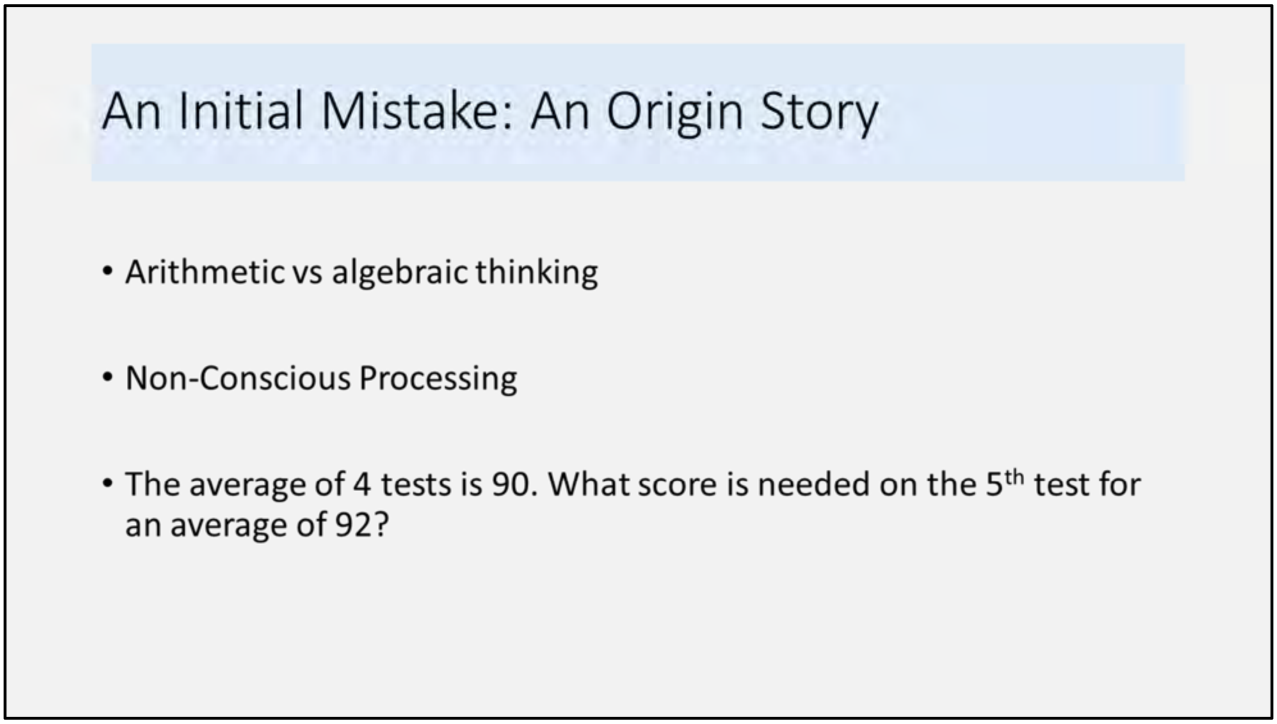
Before we get into this, let me give you a problem that you are to do in your heads. No paper or pencil and do not shout out the answer. I'll give you five seconds. Then I want to hear your answers and how you came up with it. In other words, explain your work! The problem is: 12 x 13

[We asked what people got. Most used 12 x 12 + 12 = 156. One person used the distributive method]

Possible answers to mental math problem

- $12 \times 12 + 12 = 144 + 12 = 156$
- $12(10 + 3) = 120 + 36 = 156$
- $13(10 + 2) = 130 + 26 = 156$
- $13 \times 13 - 13 = 169 - 13 = 156$

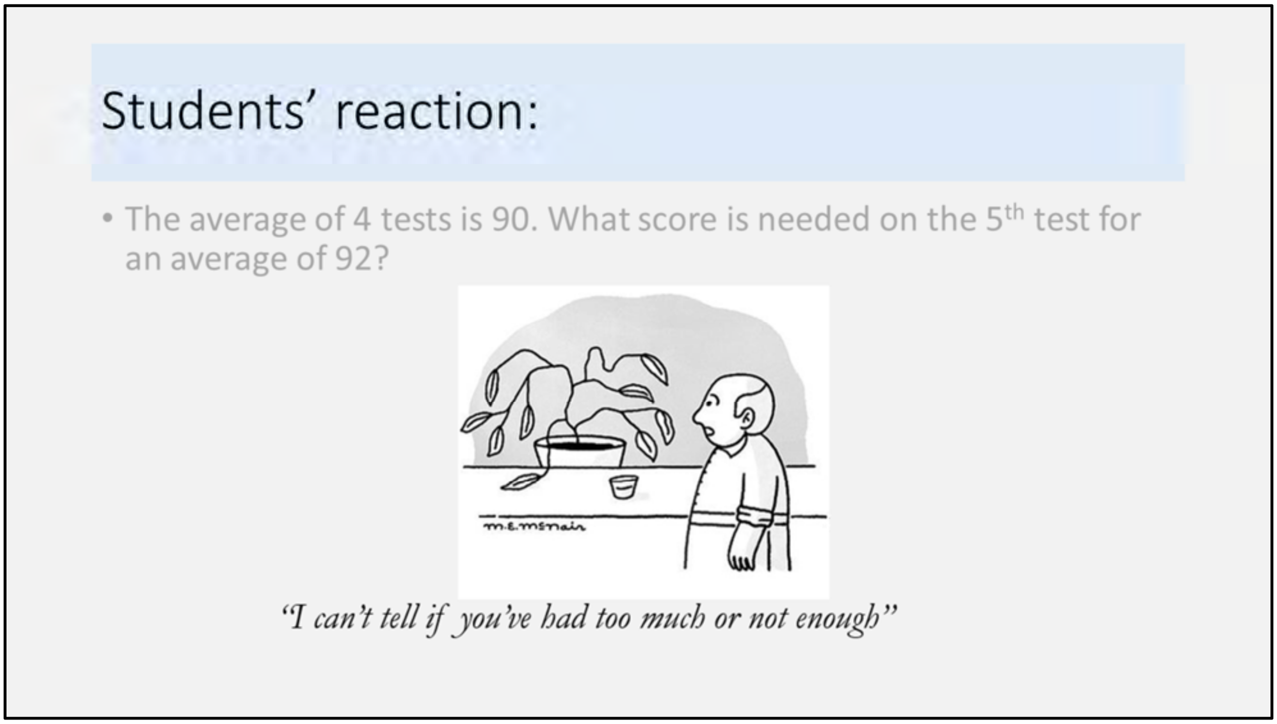
Here are possible answers. We'll come back to this later, so bookmark it in your minds. I want to start off with an origin story of my own; it's about a mistake I made when I was a student teacher and how it ultimately led me to two important component in teaching word problems:



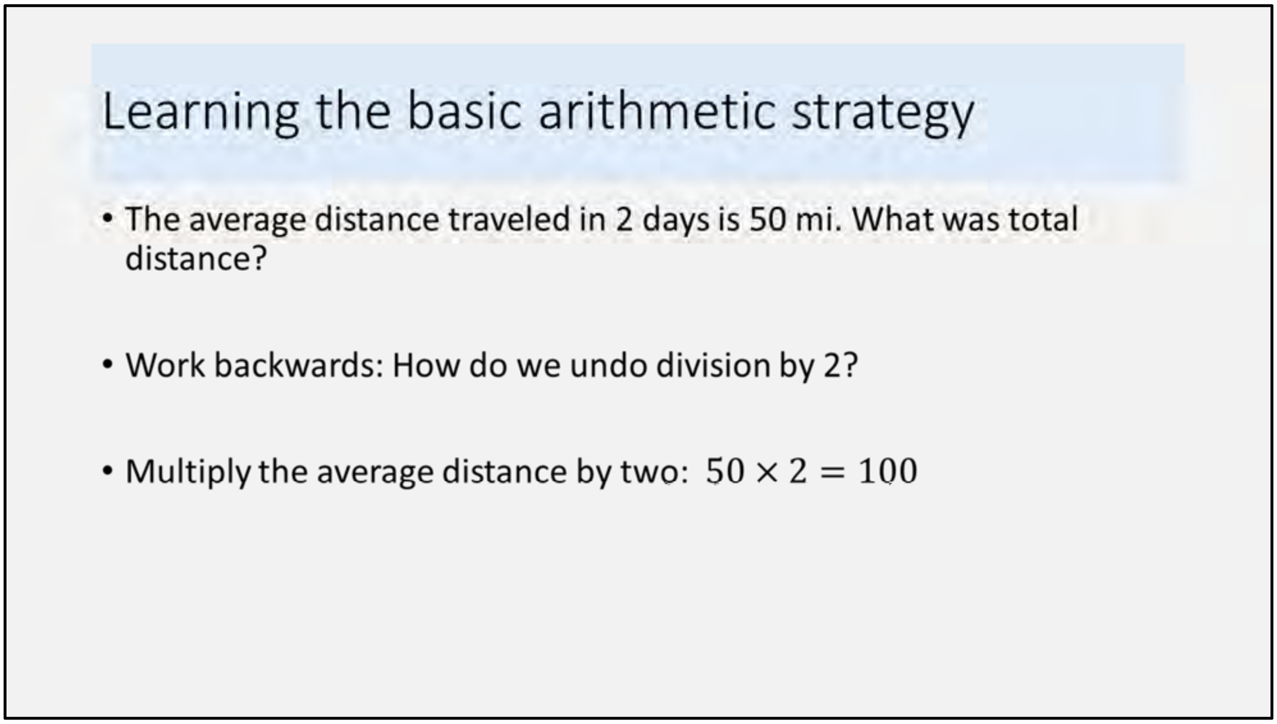
Arithmetic versus algebraic thinking

and Non-Conscious Processing. So let me talk about the mistake I made. I was a student teacher. It was the first week of school. My supervising teacher handed me a book of various "warm up" problems and told me to pick one and give it to the class

I picked the first one in the book. To make this realistic, please pretend you don't know how to solve it. And please don't shout out the answer. "The average of 4 tests is 90. What score is needed on the 5th test for an average of 92?"



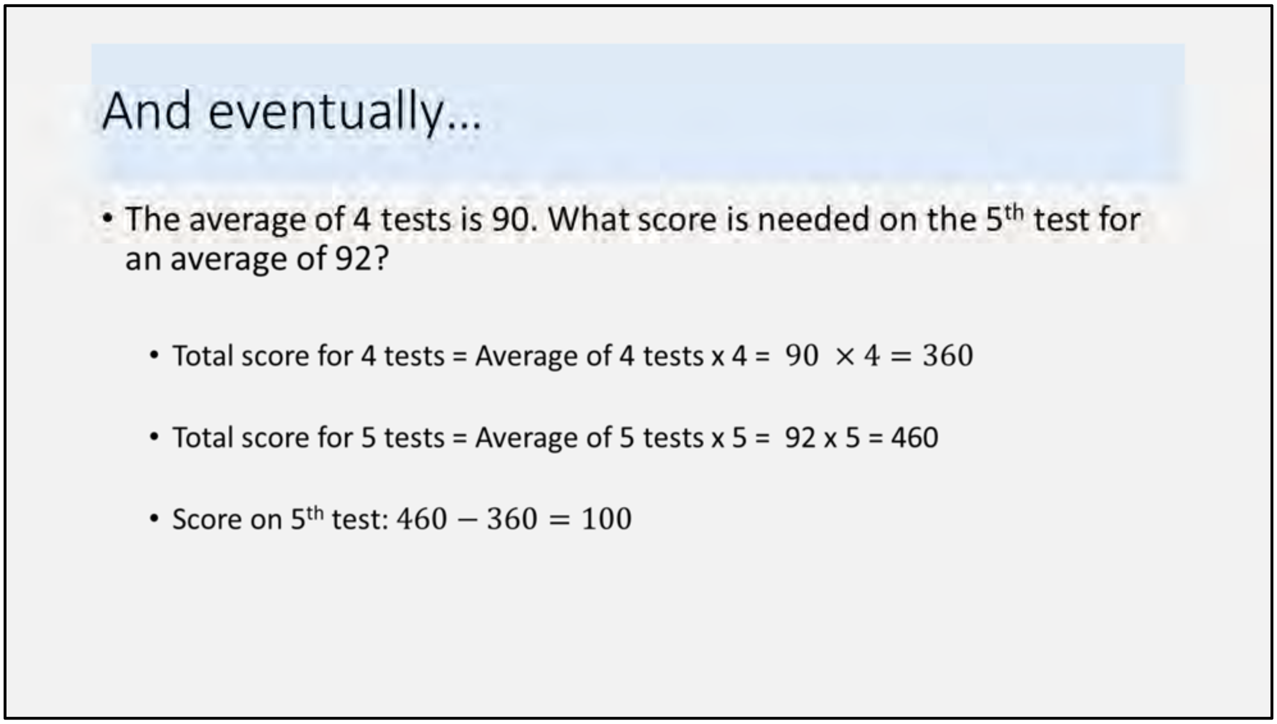
My students' response is approximated by this cartoon. Too much or not enough? Actually it was a bit of both. My supervising teacher told me later that the students needed to have simpler problems and work up to more complex ones.



When I started teaching with my own classes, I found a series of problems dealing with averages in Singapore's math books and used them as part of the day's warm-up problems; one "average type" problem each day, starting simple and ramping up in complexity. Here's the first in a series which also elicited a lot of blank stares at first.

I would ask, can we work backwards? 50 miles a day for two days? How do we undo division by 2? That usually triggered a response.

Multiply the average by 2; total is 100.This gave students an important strategy, or schema, which then provided a foundation for succeeding problems increasing in complexity…



Until we reached the problem I had mistakenly started with. Students needed some hints but the schema was in place. They saw that they needed to work with the total scores of 4 and 5 tests

Total of scores for 4 tests with average of 90 = avg of 4 tests x 4 = 90 x 4 = 360

And total score of 5 tests with avg of 92= avg of 5 tests x 5 = 92 x 5 = 460.

Some students needed a hint for the last step. Well if the total of 5 tests is 460 and the total score of 4 tests is 360 then what does this increase tell us? … I heard various forms of "OH"! Yes, the increase of 100 is the 5^{th} test score: 100.

Non-Conscious Processing

- Average score for 4 tests = 90
- My approach: Total score of 4 tests = x
- $\frac{x}{4} = 90 \rightarrow x = 4 \times 90 = 360$
- Average score for 5 tests: $\frac{360+y}{5} = 92$;
- $y = (5 \times 92) - 360 \rightarrow \ 460 - 360 = 100$

Now, this is an arithmetic approach, which is to say no algebra was used. But it brought to light how I did it when I first solved the problem. When it came to finding the total score for the 4 tests, I started with the average of 4 tests = 90.

I could have simply multiplied straight off, but I chose to find the total using an equation; letting x = the total, then $\frac{x}{4} = 90$; which leads to 4 x 90 or 360

For finding the score on the 5th test I let y = that score and came up with this equation for the average. Adding y to the total score for 4 tests and dividing by 5, equals 92.

Solving it, we end up with 460-360 or 100. Both methods end up with multiplying the average by the number of tests and subtracting the totals.

The choice of using an algebraic approach

- Using an algebraic approach was a non-conscious choice
- Many choices are made non-consciously (Sklar, 2021)
- Choices come from accumulated repertoire of knowledge
- Algebra is a generalized way of representing numbers

I was unaware that my choice of using algebra was a non-conscious one. We make such choices all the time.

Non-conscious processing is discussed in a paper by Sklar which uses the example we saw earlier when you mentally computed 12 x 13. The approach you used was a choice that was given a priority above others. Most people including me and John picked 12 x 12 + 12 for example. I considered using the distributive, but the 12 x 12 approach seemed an easier choice.

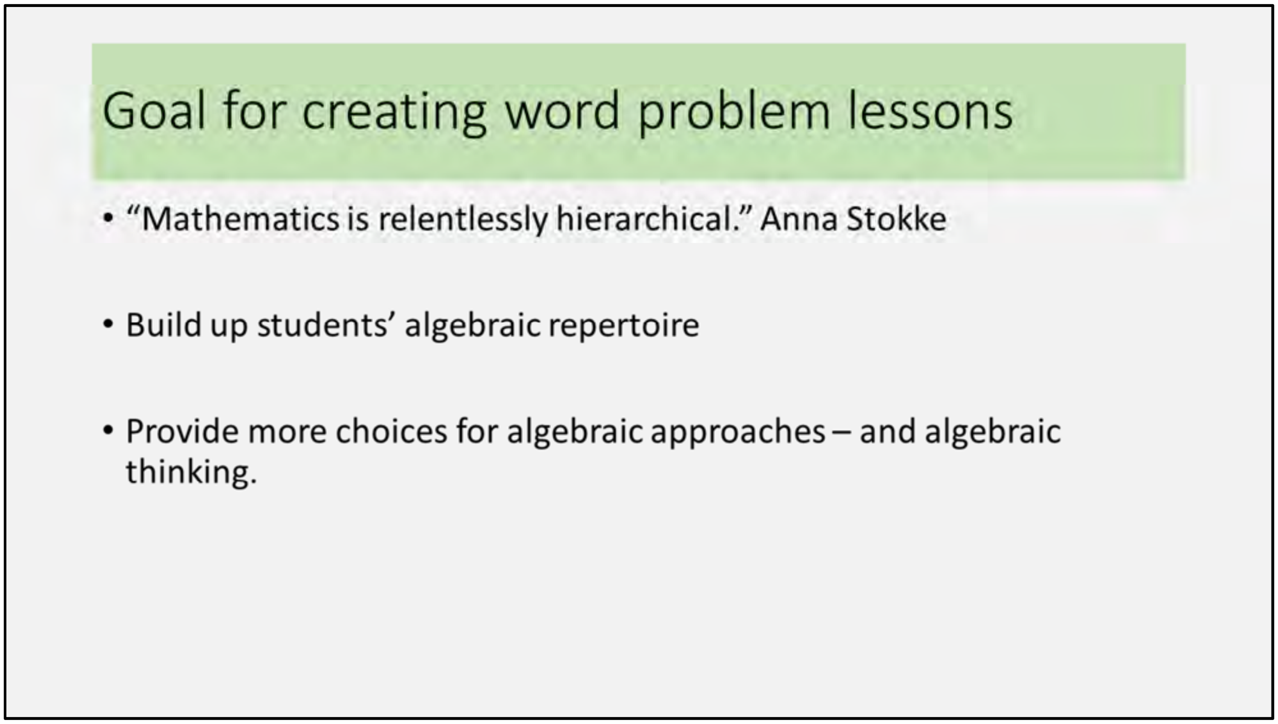
The choices come from a repertoire of knowledge. Priorities for a particular choice is largely based on experience with that knowledge. So, in the case of the problem with averages, I’ve worked with algebra for many years, and it seemed an easier way, which probably figured into the priority.

For students new to algebra, however, after years of thinking arithmetically, they are acquiring a new set of choices. Arithmetic is now becoming consolidated into a generalized way of expressing arithmetic ideas.

Reinforcing algebraic thinking

Nope, looks like we're gonna have to use the saw on this one too.

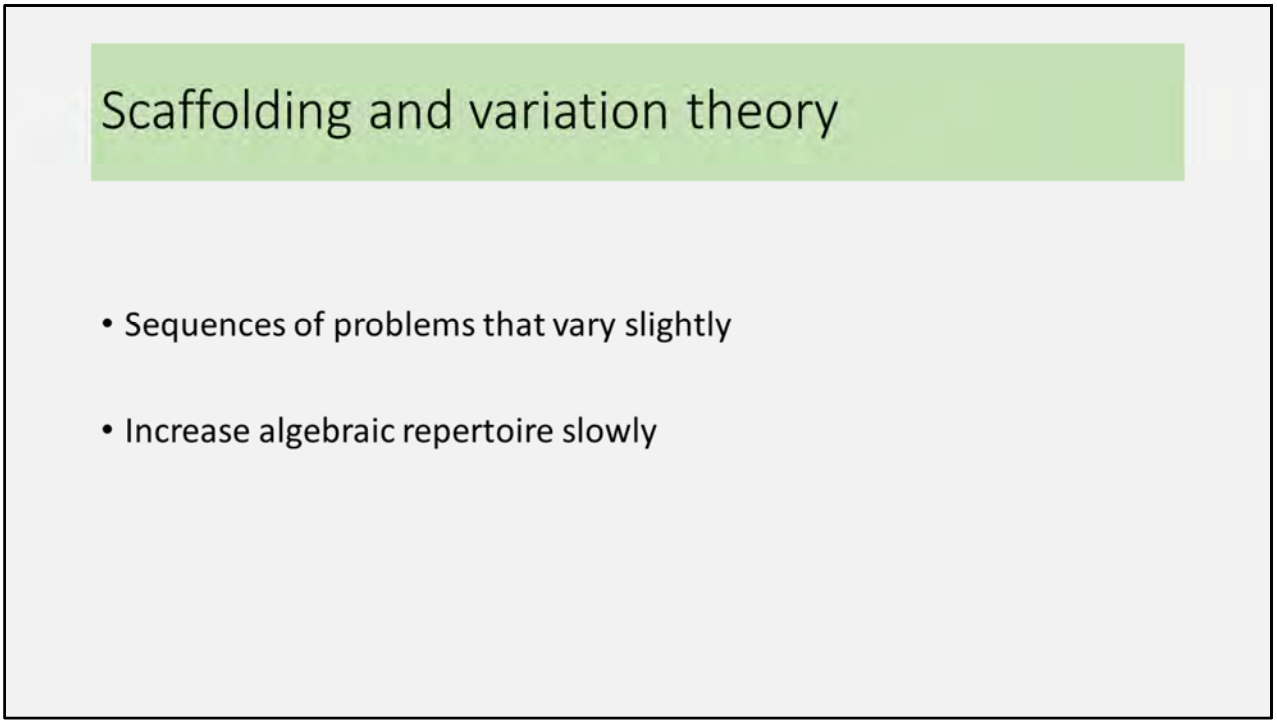
In learning algebra, students are starting to learn when to "use the saw to cut the tree down". Our goal in teaching students to solve word problems is to give the tools they need and show how to use them to solve problems. We want to increase algebraic choices, and limit arithmetic ones. John will talk about how JUMP Math works to accomplish these goals



As Anna Stokke has said, math is relentlessly hierarchical. In learning algebra, as in most subjects, students build upon their prior knowledge. But because math is so hierarchical, missing a concept or procedure compounds itself.

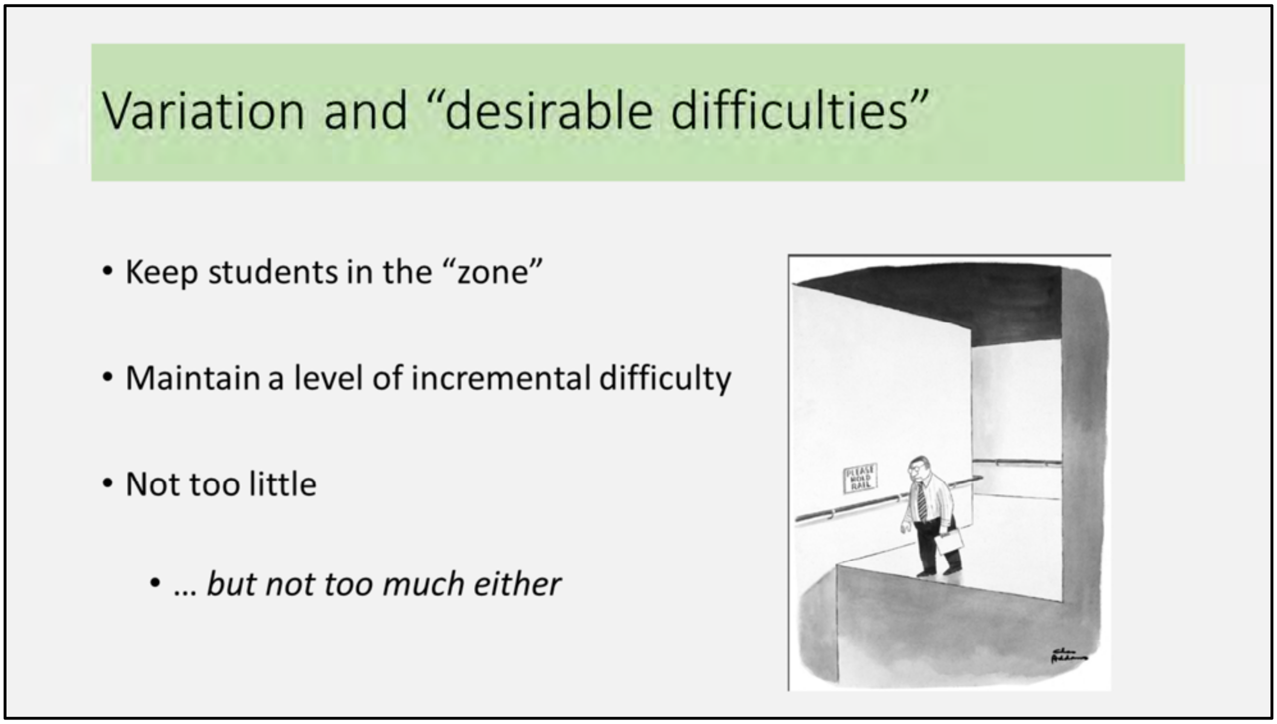
Students are continually building up their repertoire of algebraic approaches. And the principal that guides us in our lessons is to build up this repertoire a little at a time.

Ultimately in an algebra course, we are providing students more algebraic choices to draw upon. So how do we get students to think algebraically?



We do this through scaffolding; there are various ways to accomplish this. Variation theory is one way, and we use it in JUMP Math. Students are given sequences of problems that are similar, but maybe one item in the problem is varied, adding to the complexity a little bit at a time.

In this way, the student's mastery of various algebraic concepts and procedures is increased slowly as is their repertoire. Students are given worked examples which are then built upon by the succeeding problems each of which is slightly more complex

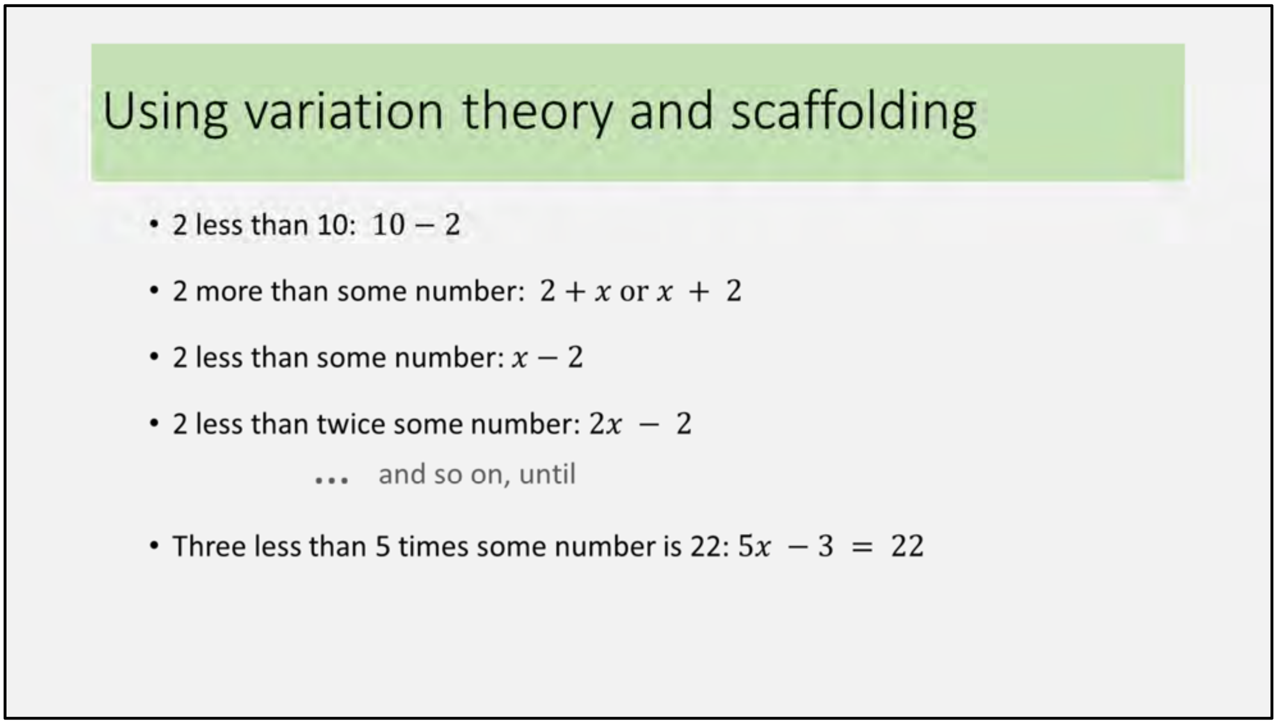


In so doing, JUMP Math keeps students in the "zone"

We do this by maintaining an incremental level of difficulty – which we call a "desirable difficulty". Students are challenged, and now must reason with what they know to solve the problem. This contributes to their feeling of success, motivating them to continue.

That said, we don't want the increments to be so small that students ask "where are we going with this?

But we also don't want too large a variation in difficulty and make the challenge so great that they want to give up.



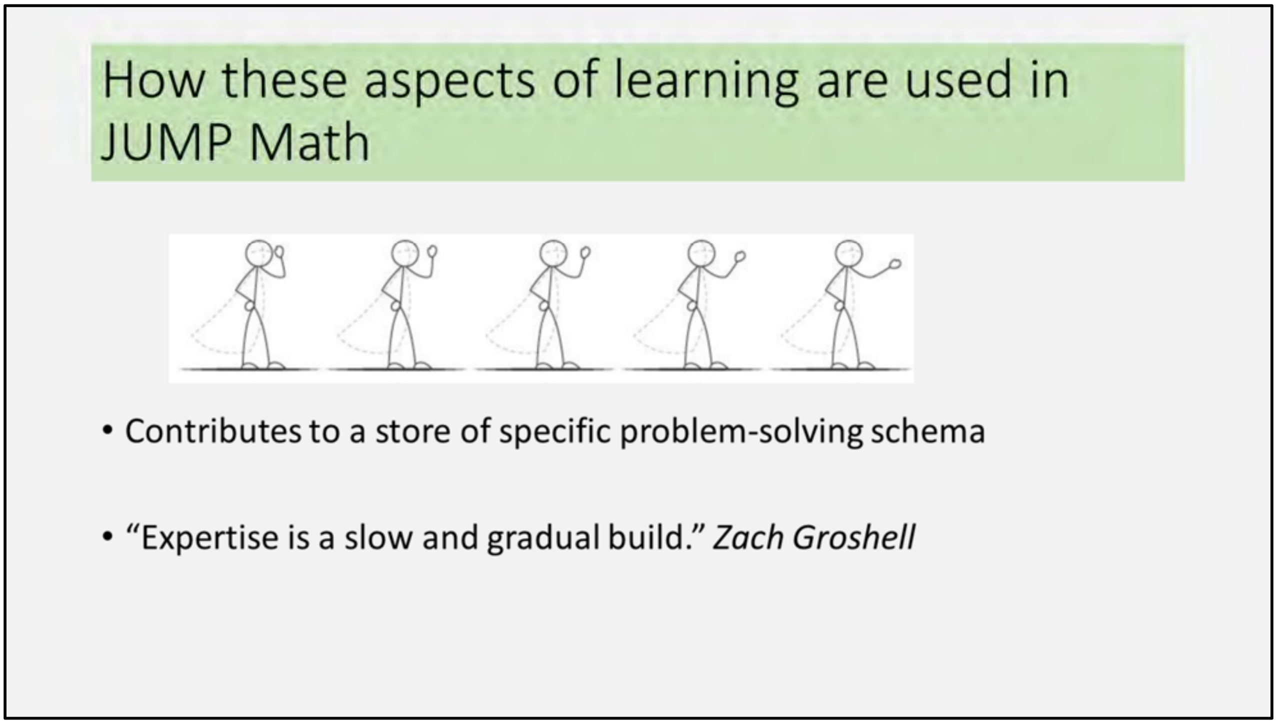
So how do we do this? Here's an example of how we design a sequence for translating English to algebra. We start with an arithmetic example of writing "2 less than 10"

And we want to end up with an equation: "Three less than 5 times some number is 22." So what would be the steps that come in between to get there?

To give you an idea, here are some. These can vary. Students have no problem writing 10-2. We then introduce variable expressions; they know that "some number" means a variable. "Two more than some number" translates to 2 + x

Next: 2 less than some number. Many will say 2 – x. We ask them how they wrote 2 less than 10? So they see with subtraction, the order is important: So we have $x - 2$.

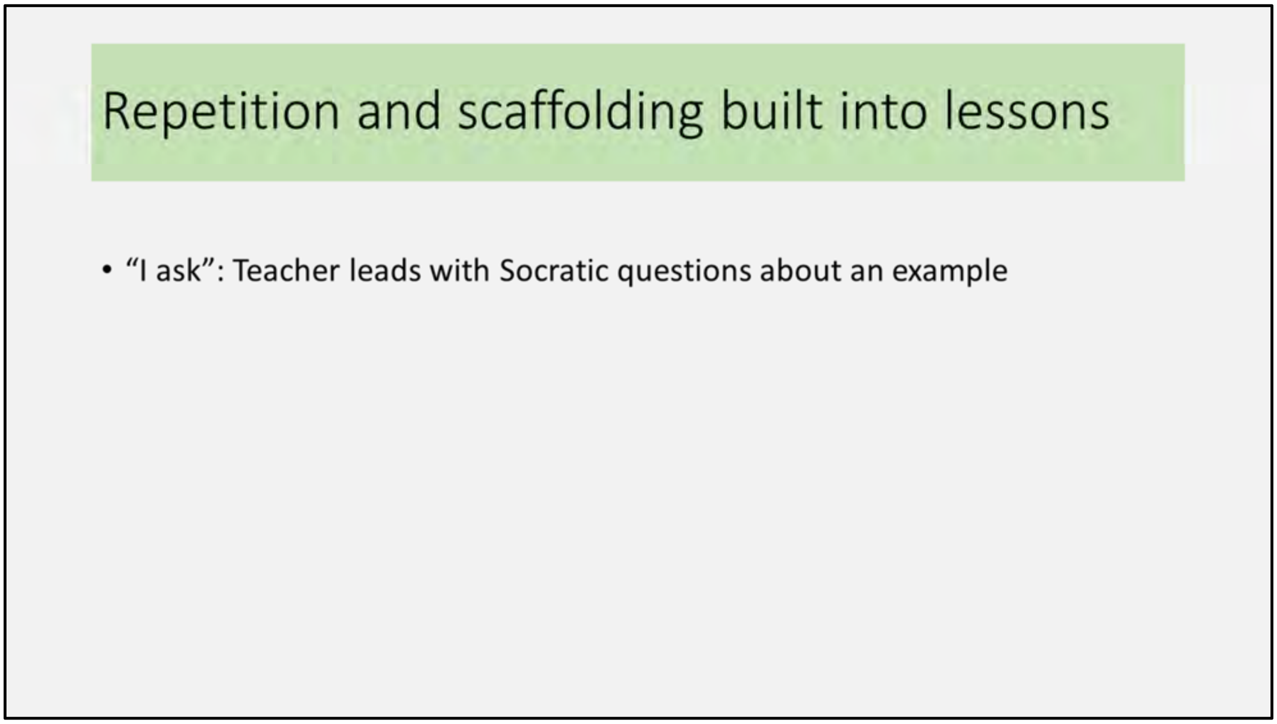
And so on … until we reach the equation we wanted to get to, translated to 5x-3=22.. This is the approach used in JUMP. Sometimes the increments are slight like we just saw; other times, there are larger steps between the variations



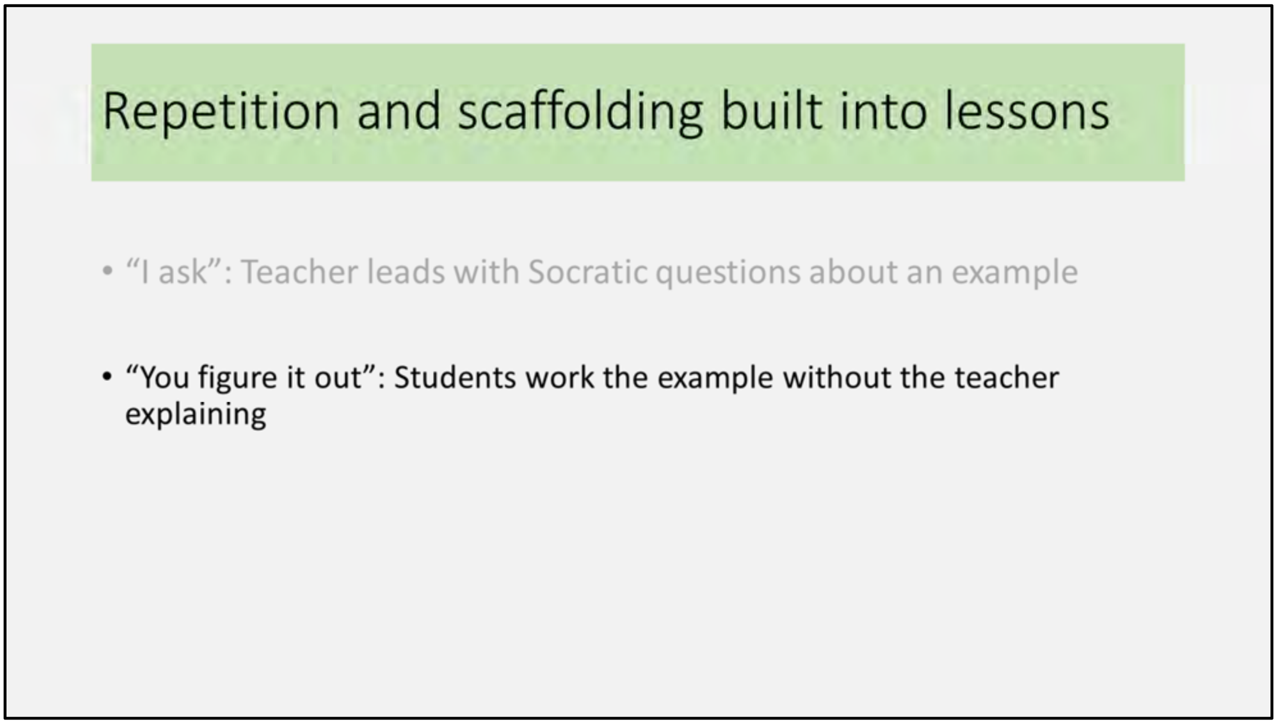
The scaffold/variation theory approach in JUMP is similar to animation and the actions that happen between the frames of a movie or a cartoon. In animation we have "persistence of vision". In algebra we have "persistence of concept".

The acquisition of incremental knowledge contributes to a store of methods, concepts and specific problem-solving strategies.

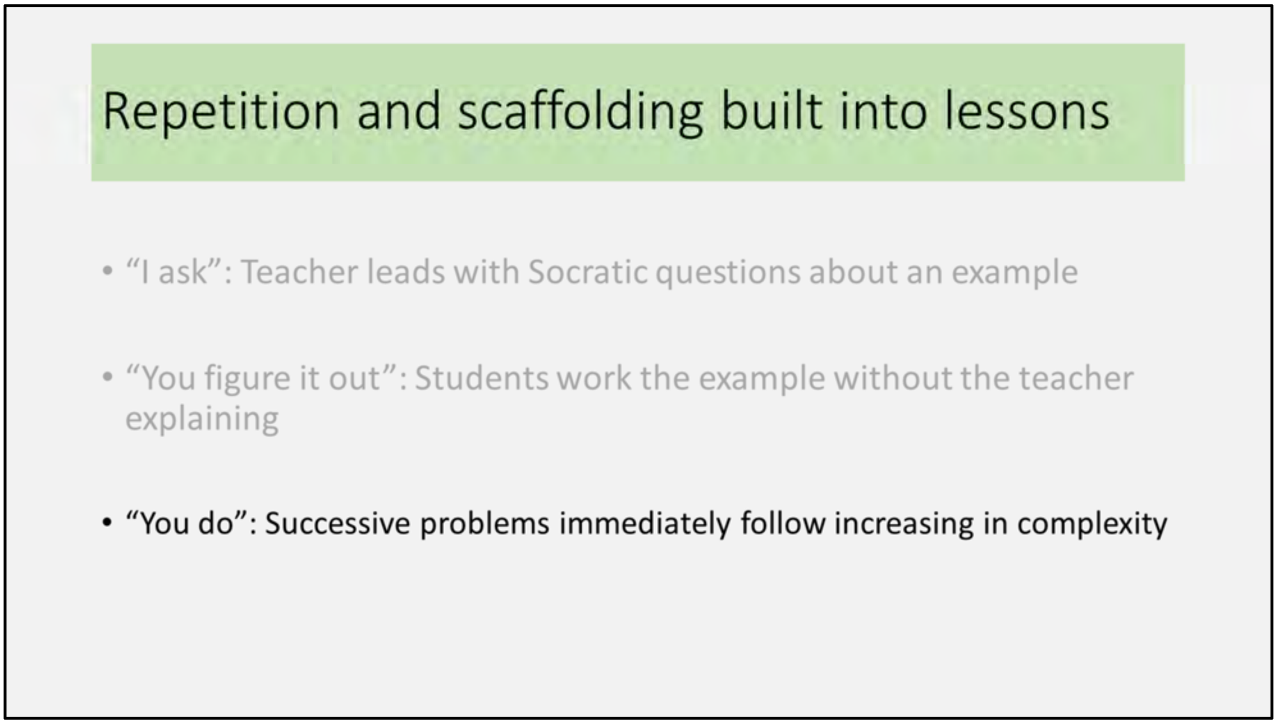
Students are progressing and mastering a little at a time. To quote Zach Groshell: "Expertise is a slow and gradual build."



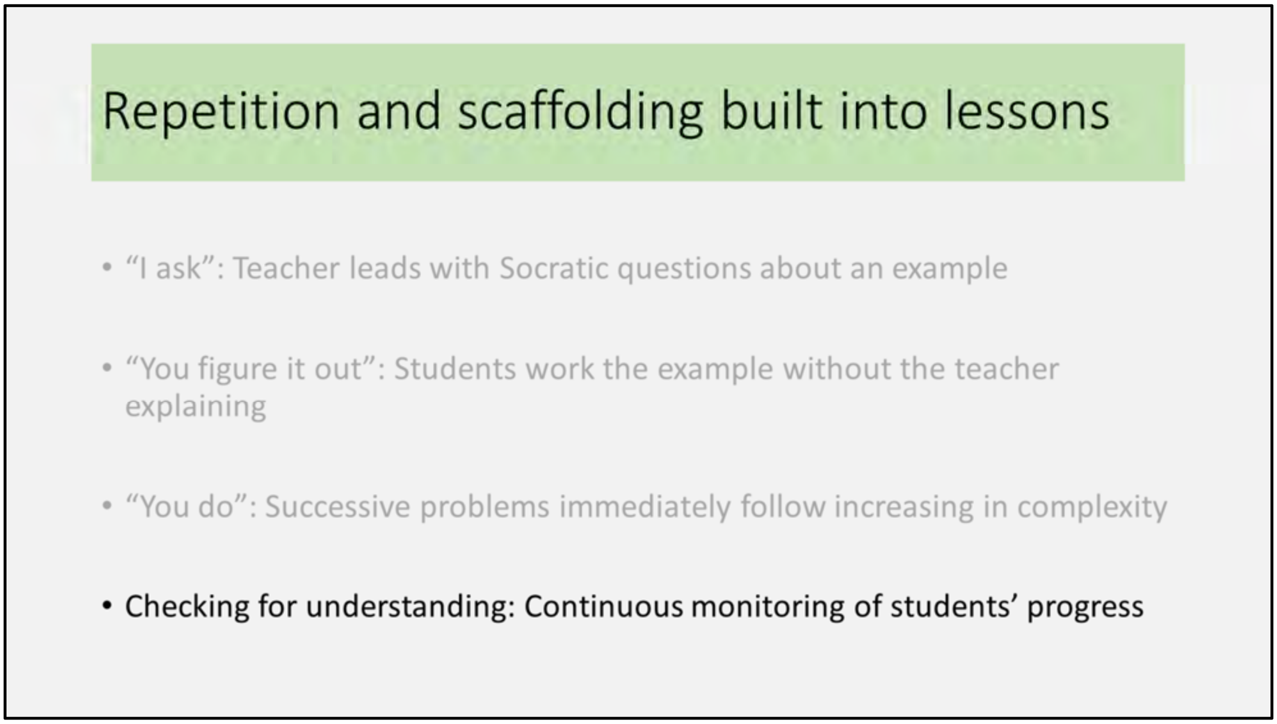
JUMP Math’s approach uses digital slides for each lesson, each slide containing a problem or question. JUMP often uses a variation on the “I do, we do, you do” method of explicit instruction. We call it “I ask, you figure it out, you do.” In the “I ask” stage, the teacher leads with Socratic questions about an example



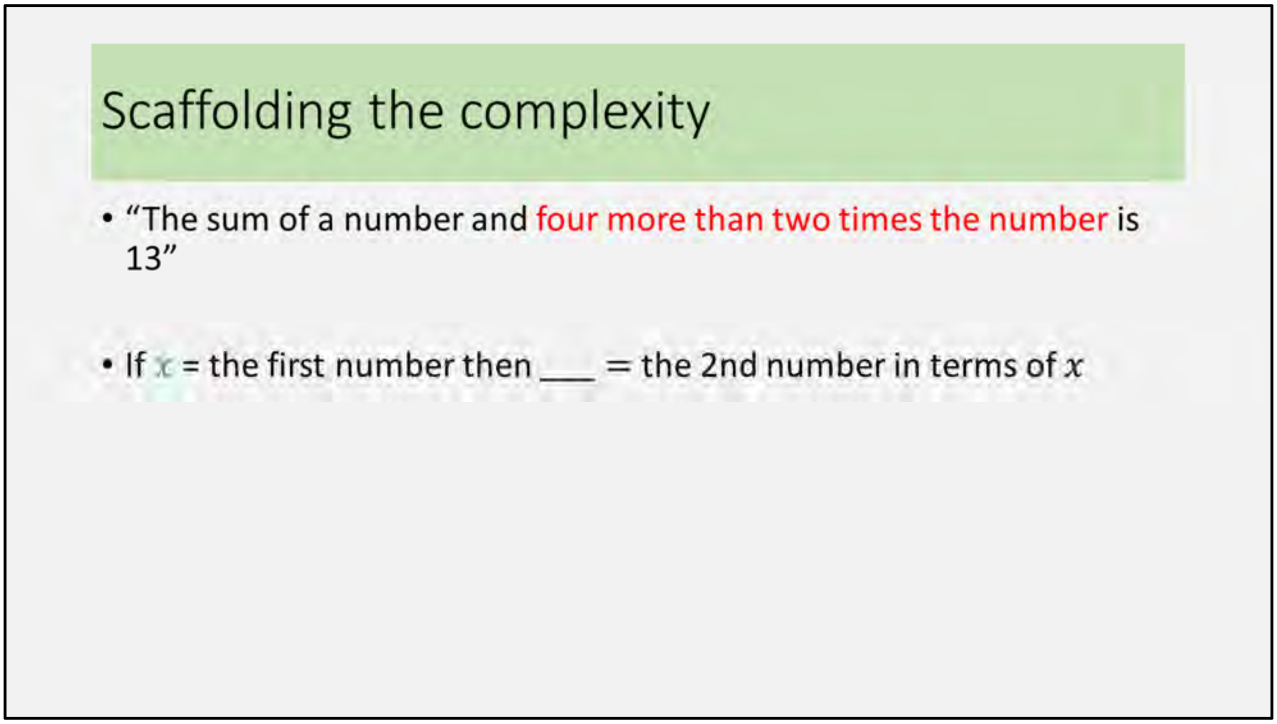
In the "You figure it out stage" students see what to do next, essentially working the example with little or no explanation from the teacher. Students make the connections themselves.



In the "You do" stage, successive exercises immediately follow that increase in complexity and a fading of support

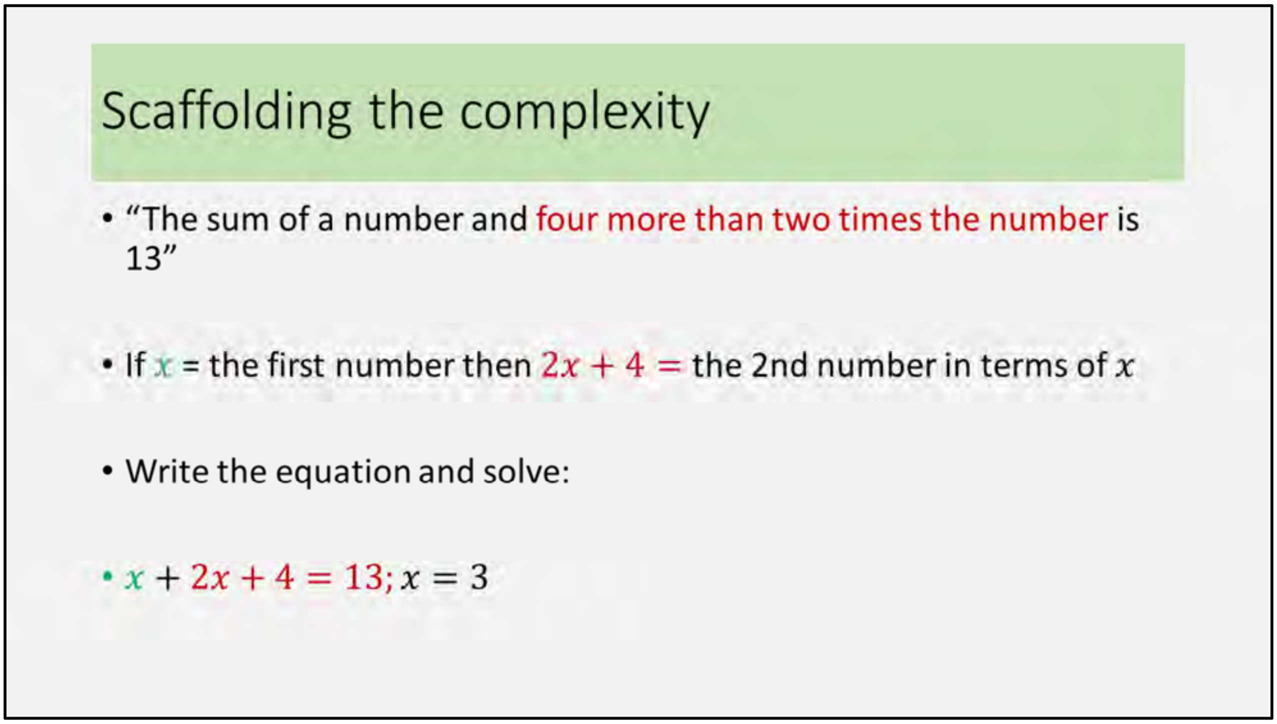


In this stage teachers are continuously “checking for students’ understanding” monitoring progress and providing feedback and guidance as needed. When the scaffolding is done well, it is easy to spot those students who need additional help and guidance.



Now let's see how we progress from translation into algebra to solving some basic word problems. They now are given a problem like this, with two unknowns: "The sum of a number and four times the number is 32." The two unknowns are to be defined in terms of one variable.

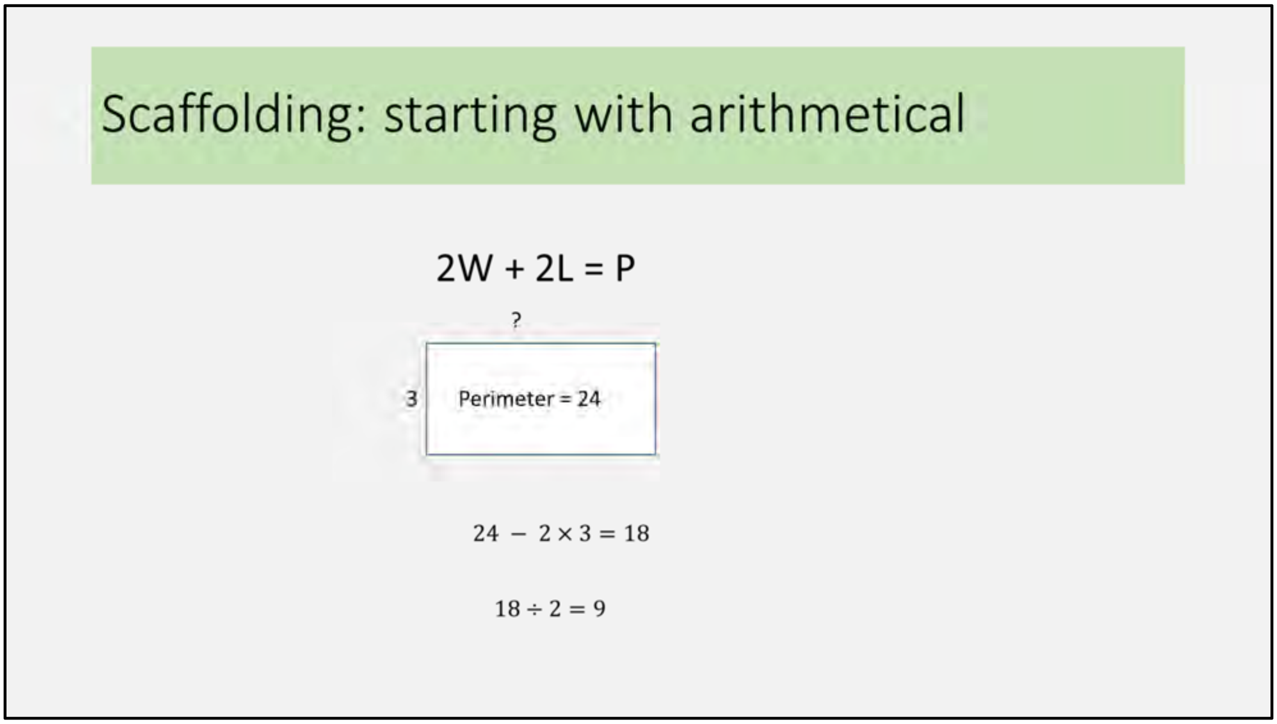
They are then given a prompt in which the student is told that x represents the first number. They have to express the 2nd number in terms of x. They've had practice translating English into algebra. Now they retrieve that knowledge and put it to use.



They fill in the blank: 2x + 4

They're prompted to write the equation and solve.

The first numbers is x, and the second number is 2x+4; the sum equals 13. They then solve it.

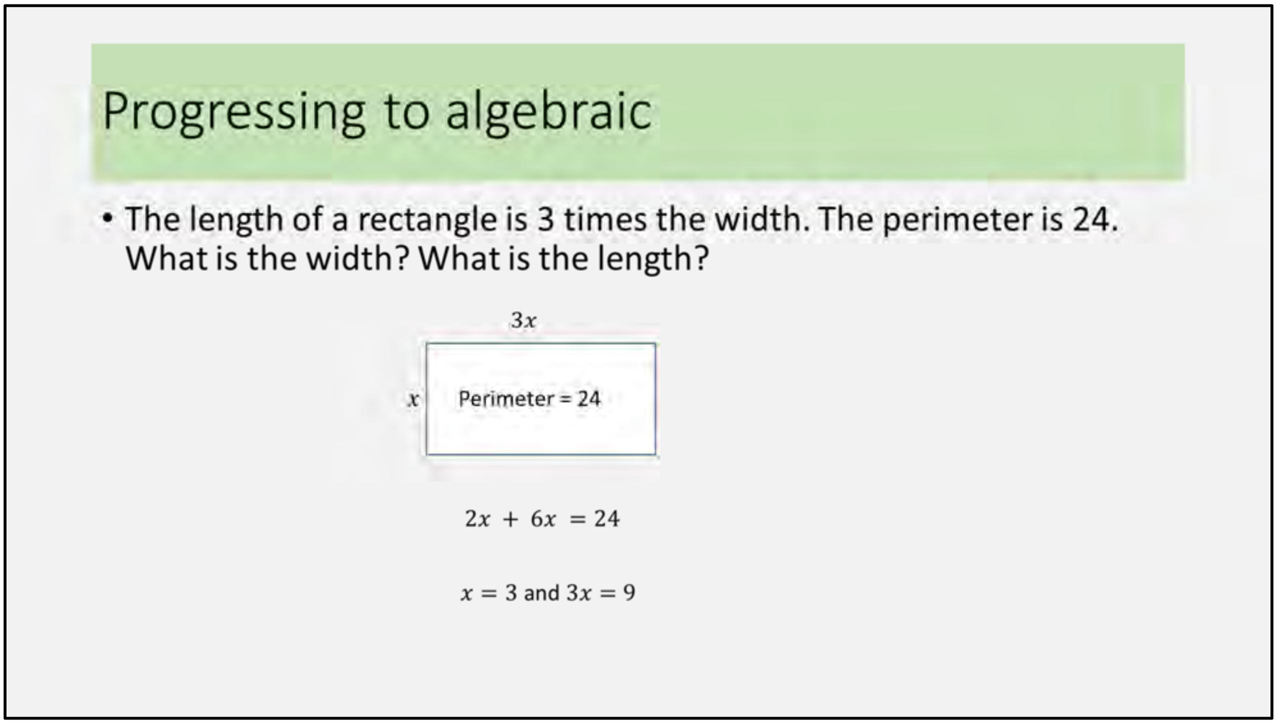


Now we move on; students will now apply what they've learned working with numerical problems with more than one unknown to problems dealing with many situations but let's take perimeters of rectangles as an example. We start with arithmetical approaches, which students have worked with in 6^{th} and 7^{th} grades. We first remind them of the formula for finding the perimeter of a rectangle.

Then we have them solve a problem; a rectangle'; with a given perimeter of 24; and a width of 3 units; has what length?

Most students will do this in their heads; maybe subtract 2 times the width from 24. Others might make use of the formula and write an equation.

Some might stop at 18 as the answer in which case we ask "Is 18 one length or the sum of two lengths?" They'll then divide by 2, and we have the answer. It's 9.



Now we introduce similar perimeter problems but with variables, so students now are required to use an algebraic approach. These problems have the same structure as the numerical problems they just did, but what looks obvious to us will not always be so for novices; this one will serve as a "you figure it out" worked example.

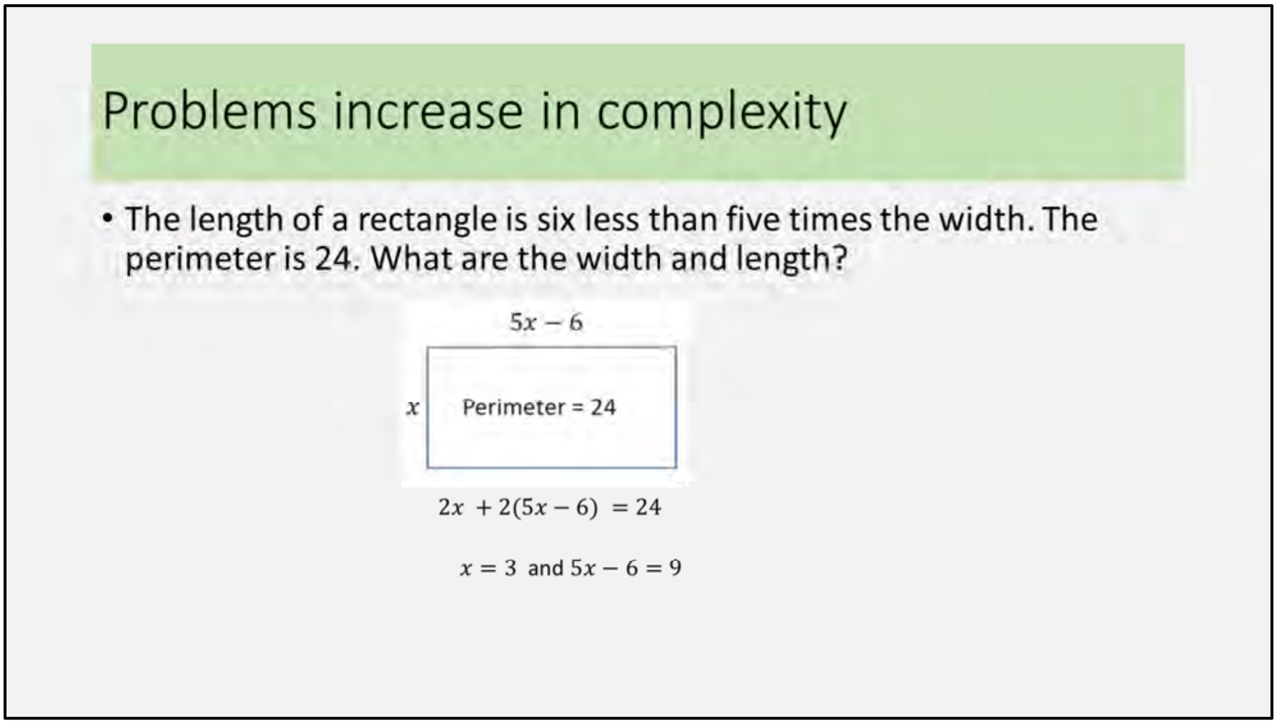
Like before we have the rectangle, the perimeter is the same as before: 24.

Students are told x represents the width; …but are then asked to figure out what the length would be in terms of x. And it's 3x which they know from their practice with translating into algebra.

They know the formula for finding the perimeter; they are prompted to plug in the variable expressions for width and length

They state the equation., Two times the width (2x) plus 2 times the length (6x) = the perimeter of 24.

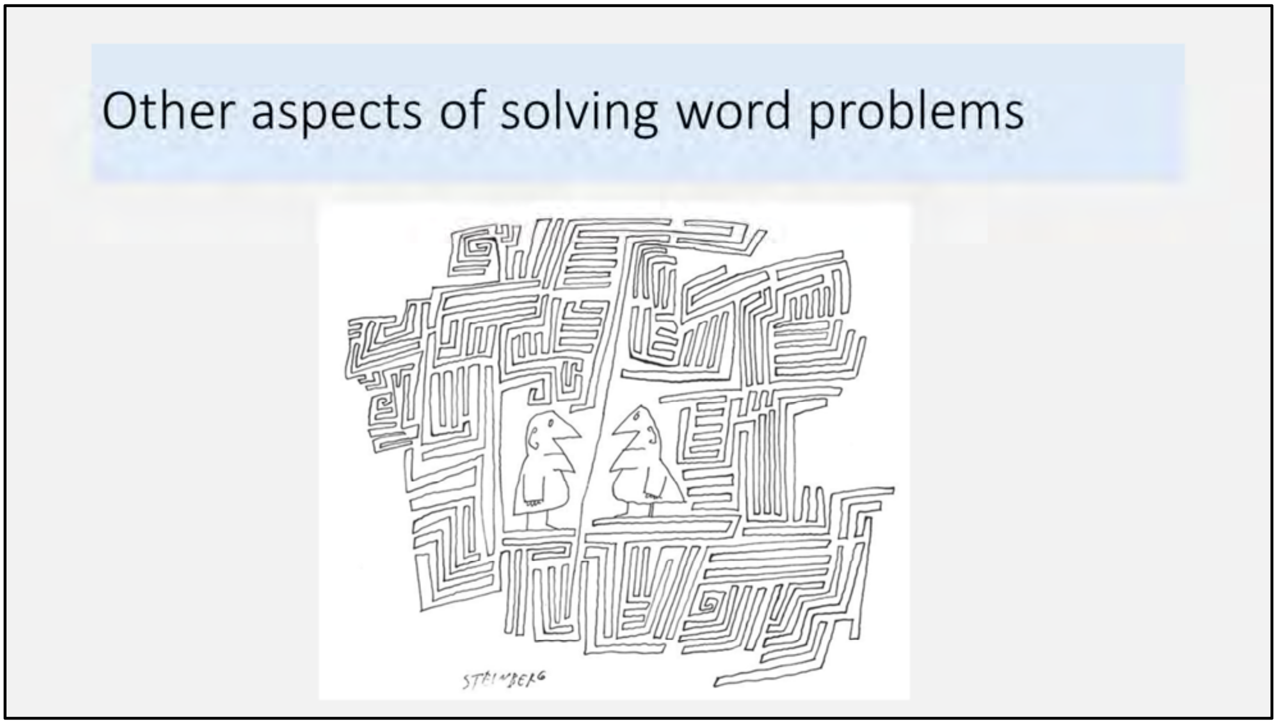
and solve it. √ Width = 3, plug that in to 3x and we get the length = 9
… And we continue to scaffold the problems



The problems become more complex, but the structure of the problem is the same. We now ask the student what we let x equal. Based on the worked examples preceding this problem, they see letting x = width is the easiest way to go. Now the expression for length in terms of x (the width) is: $5x - 6$.

The equation then follows; 2 times the width (or 2x) + 2 times the length (or 2 times (5x-6) = the perimeter (24).

We get width = 3, and plug 3 into 5x -6 to get length = 9. Now Barry will describe how we approach more complex problems as well as some aspects of problem solving in general



After working with the basic type of word problems we've just seen, we move on to more complex ones. These involve "what to do with what" in solving word problems.

Solving problems doesn't happen on its own

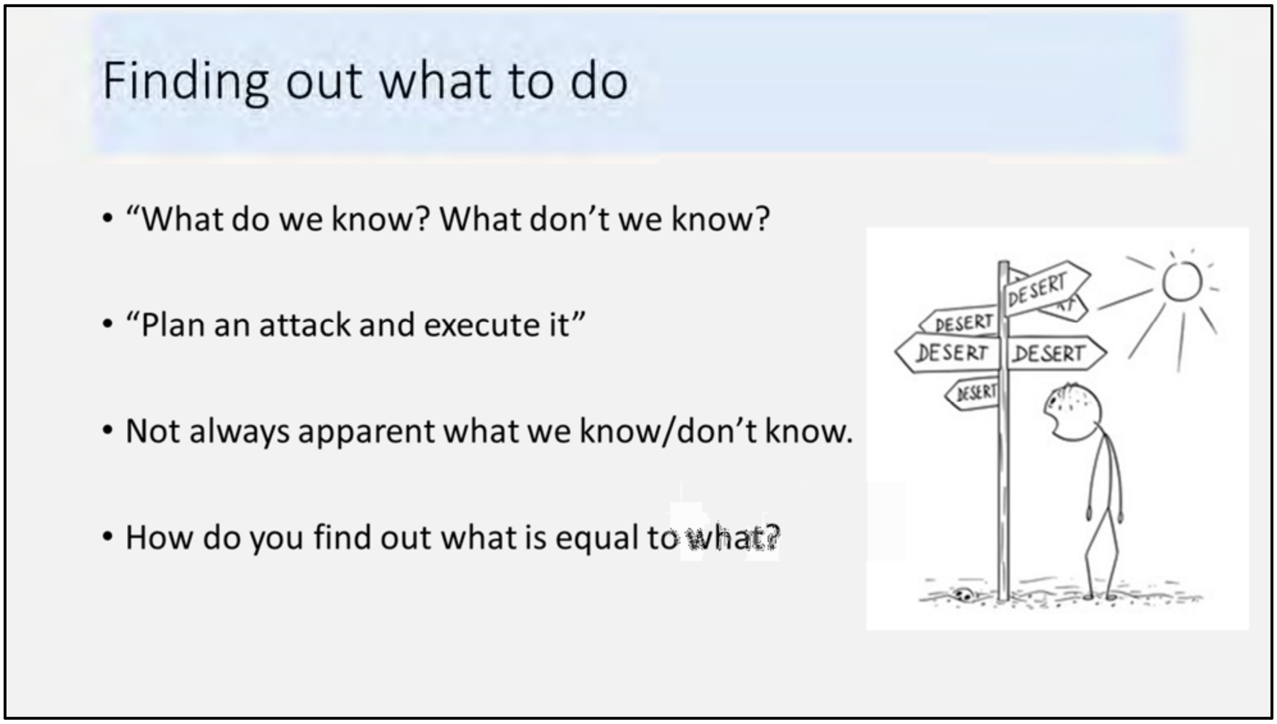
- Learning to solve word problems is domain-specific
- Teaching general problem solving skills does not lead to math skills or knowledge. (*Sweller, Clark, and Kirschner; 2010)*
- Imitation is the first step in learning anything new.
- Reasoning with what's imitated then follows.

Solving problems doesn't happen on its own. Learning to solve problems is domain-specific. Some believe, however, students should use general problem-solving skills, and apply it to problems for which they have been given little to no background and for which the connection to previous problems is not apparent.

But it really doesn't work that way. As Sweller, Clark and Kirschner discuss in their paper on problem solving, teaching generic problem solving skills does not lead to mathematical skills or knowledge; nor is it effective to learn such skills on a just-in-time basis. Yet the idea persists that explicit instruction and scaffolding are not "authentic" learning. Rather, students are held to be "mimicking" problem solving steps without thinking or knowing what they're doing.

In fact, imitation is the first step in learning anything new. As anyone knows who has tried to learn an instrument, a dance step, a language, a golf swing – imitation is harder than it looks. This applies to math as well.

In properly structured lessons and course of instruction, students use what they've imitated as a tool – a tool with which they reason to solve more complex problems.



I sometimes see a checklist of problem solving steps in textbooks, that amount to generic type skills. Some things like "Find out what you know and what you don't know…

And plan an attack and execute it; on it's own, these are not bad pieces of advice, but…,

… some things that students need to know are not stated in the problem, as we'll see shortly. And even so, students still need to be taught what to do with what they do know in order to "plan an attack".

That means **finding out what is equal to what in a particular problem.**

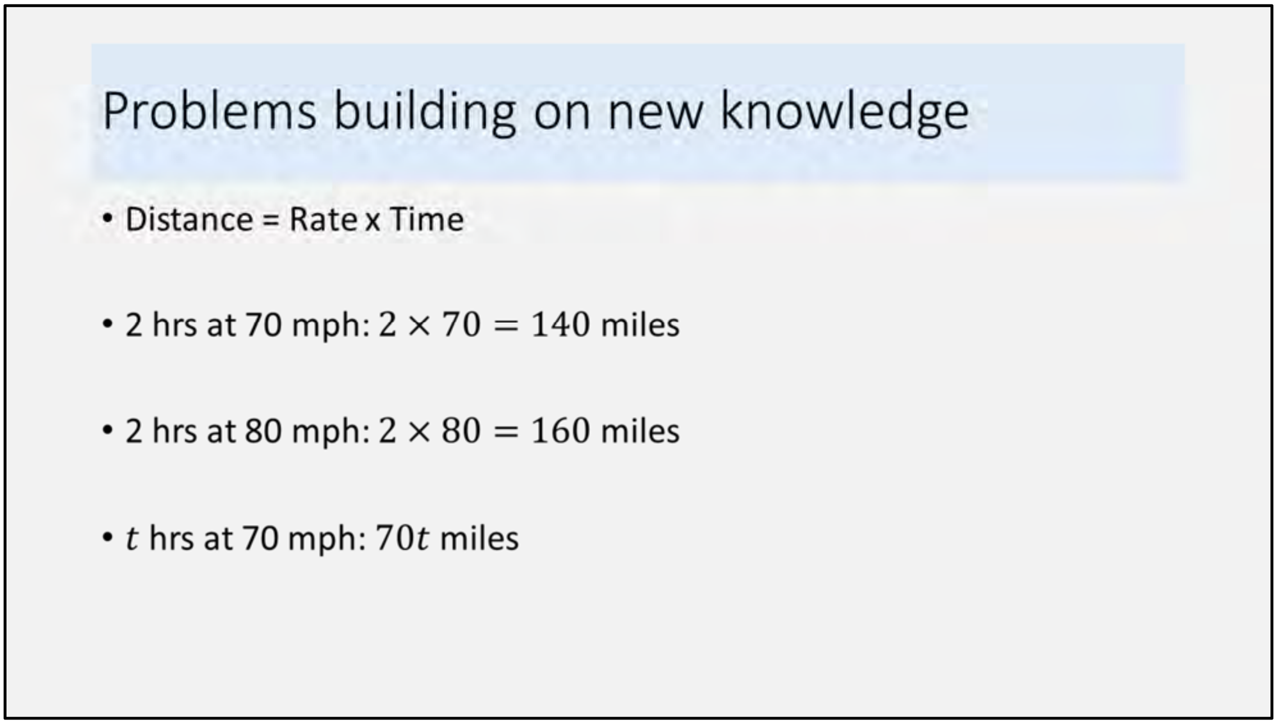
When students are deprived of necessary instruction or guidance on how to do this it can leave them feeling like this guy

Finding out what to do (cont)

- "The sum of a number and three times the number is 24".
- "Two cars 600 miles apart head toward each other. Their speeds are 70 and 80 mph. How long will it take for the cars to meet?"

As we've seen, for beginning problems such as this one, it becomes obvious to students what they know and don't know and what is equal to what. But pretty soon, we come to more complex problems where it may not be so obvious.

For example, this distance/rate problem. It's a pretty big jump from problems where it's obvious what things are being equated. For problems like this the basic form is distance = distance. So let's look at what they need to know for motion problems.

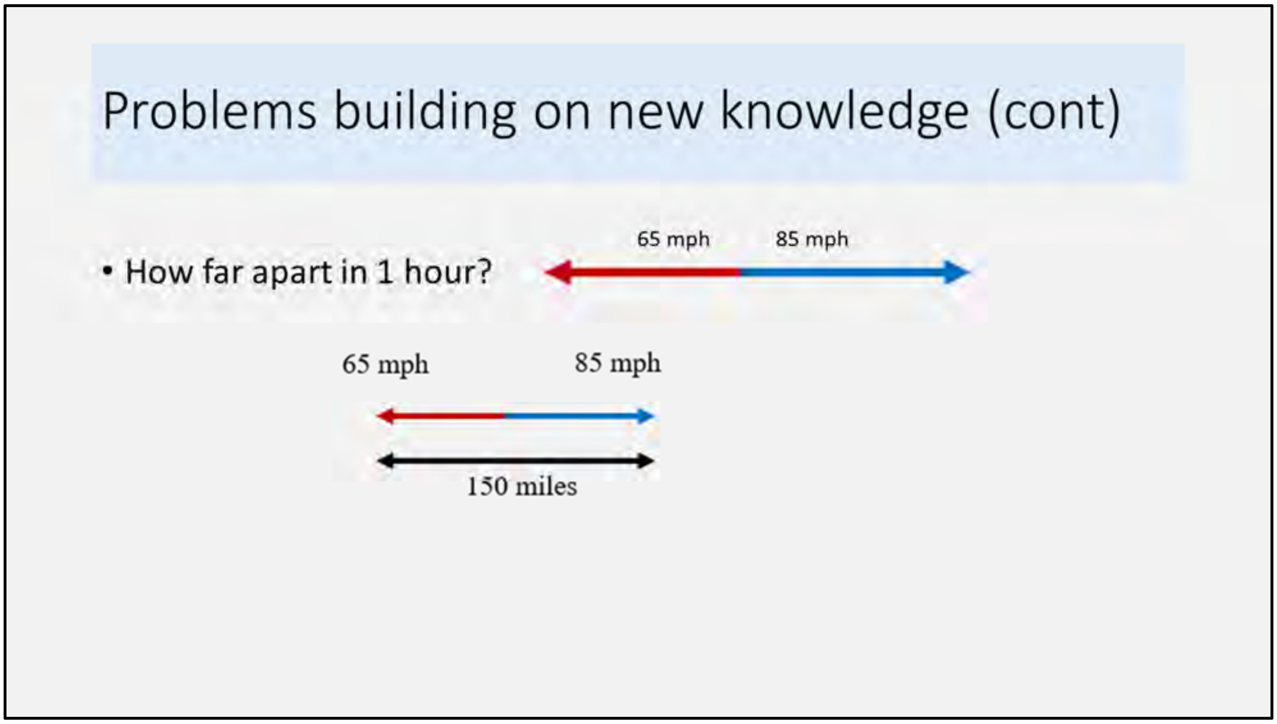


What they need is some basic information. In this case the formula: Distance = rate x time is our starting point.

Then some examples. If a car travels 2 hrs at 70 mph what is the distance?

2 hours at 80; maybe a few more to get them comfortable and fluent, and then more like that, 30 mph for 5 hours, 70 mph for 10 hours, and then …

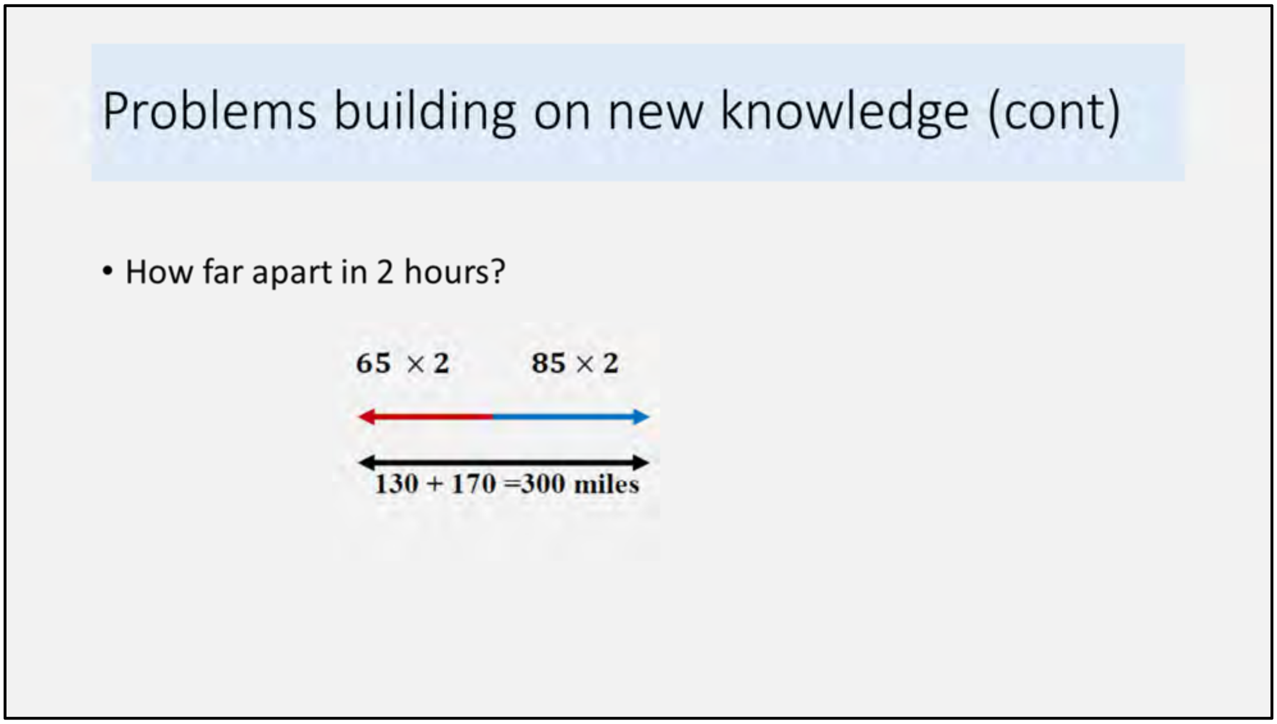
…70 mph for t hours. Students see, maybe with some prompting, that it's 70t. More like this so they get used to it, and then we're ready to apply this to a problem. But instead of a "traveling toward each other" problem, I start with a "traveling away" type problem.



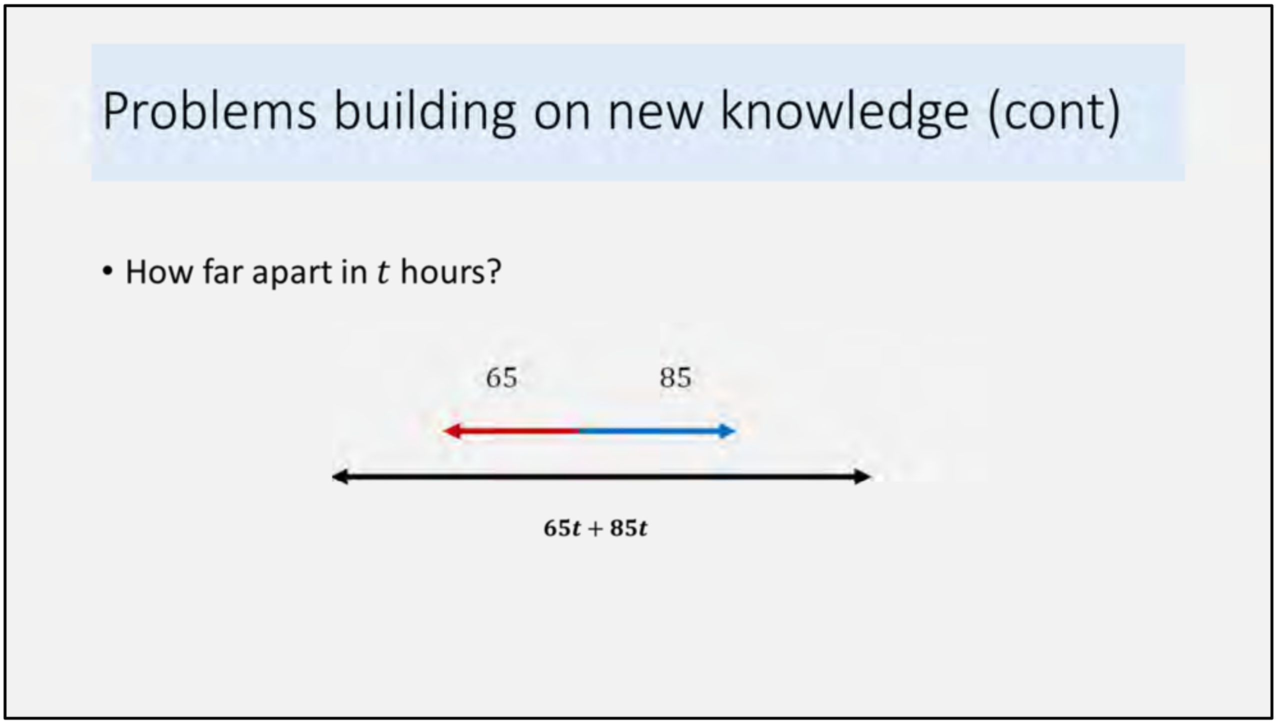
We begin with two cars traveling away from each other, starting from the same pt at the same time, at 65 and 85 mph.

We ask: How far apart will they be in 1 hr? With some prompting students see "what is equal to what"; the sum of the two distances equals total distance and they add the respective distances to get 150 miles.

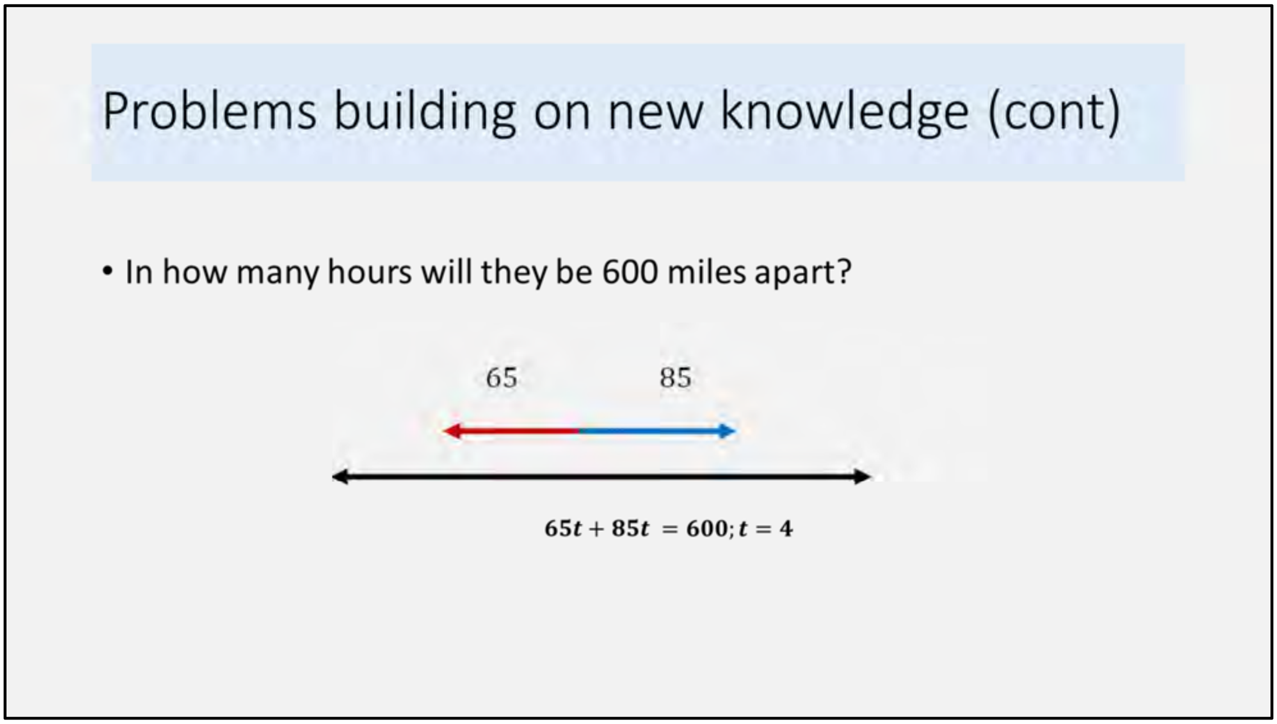
And now they figure the distance apart after 2 hours, following the same pattern. And then a few more varying times; 3 hrs, 4 hrs, so on…



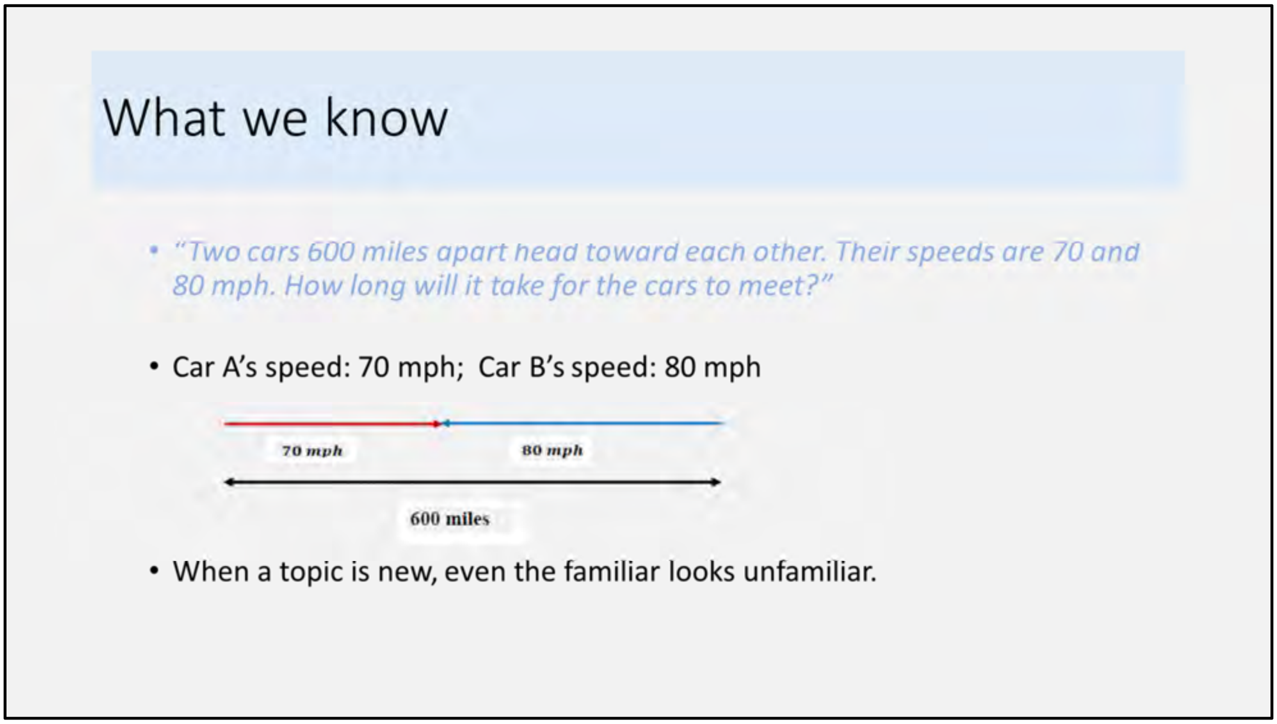
…and then a problem asking how far apart after t hours. They follow the pattern from the previous problems and see that it's 65t + 85t. And then…



…and then a problem asking how far apart after t hours. They follow the pattern from the previous problems and see that it's 65t + 85t. And then…



… how long will it take to be 600 miles apart? They know that after t hours they'll be 65t + 85t miles apart. And I'll ask what does the problem say is the sum of the 2 distances? It says the sum is equal to 600, and that leads to an equation which they solve. They do a few more of these to get the feel of them. My experience is that students catch on fairly quickly to the traveling away problems since they tend to be intuitive.



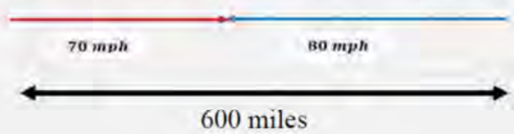
After they are comfortable with those problems, we move to the problem of two cars traveling towards each other. These are a different kettle of fish.

Here's what we know. The speed of the two cars: 70 and 80 mph, driving towards each other; 600 miles apart. This is really the problem they just solved but in reverse! The concept of sum of 2 distances = total distance is the same for both; and the problems are solved the same way. For some students, however, it seems like an entirely different problem.

When a topic is new, even the familiar can look unfamiliar. But students will soon see that the same concepts are at play in both types of problems.

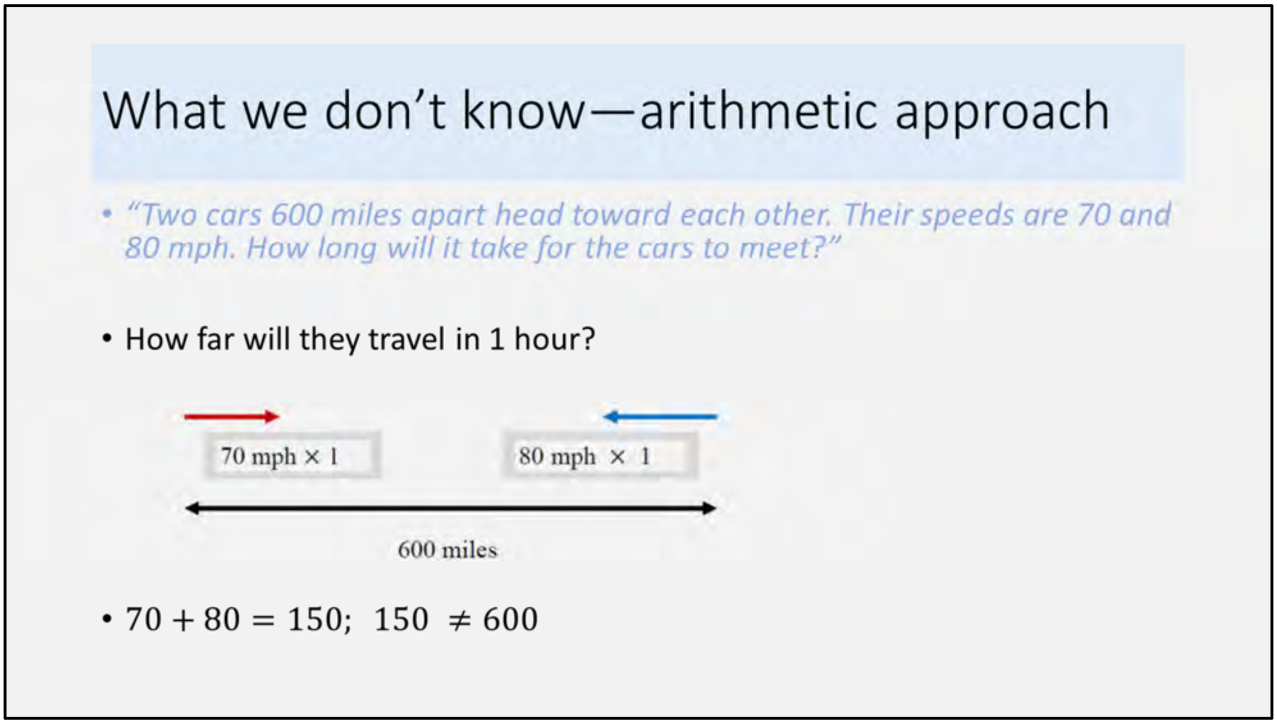
What we should know but is unstated

- *"Two cars 600 miles apart head toward each other. Their speeds are 70 and 80 mph. How long will it take for the cars to meet?"*
- They both travel for the same amount of time.
- Sum of their respective distances = Total distance



What do students need to know but is not stated in the problem? First is that the time for the two cars to meet is the same for both. This concept seemed obvious in the "traveling away" problem; it's obvious in this one but for some students it isn't. The familiar is looking unfamiliar. I've had students ask: Wouldn't the faster car get there first? One way I've showed this concept is by having two students a distance apart walk toward each other, one fast and one slow, starting at the same time. They see that they both meet in the same amount of time.

The 2nd thing not stated is that that the sum of the 2 distances = total distance. The demonstration with the students shows that their respective distances until they meet equals the total distance they were apart, which connects to the "moving away" problems. And in terms of the "moving towards" problem, the sum of the distances each car travels to their meeting point = 600 miles.



Now to solve the problem. I first use an arithmetic approach: guess and check. Cautiously! (I'll explain more in a minute.) So we start with finding how far they travel in 1 hour.

That's fairly easy: 70 x 1 and 80 x 1, and they've traveled a total distance of 150 miles. If the cars met in one hour the sum of the 2 distances would equal 600. But 150 does not equal 600; so they each need to travel a little longer.

Moving right along…

Time	Car A Rate	Car A Distance	Car B Rate	Car B Distance	Total	= 600?
1	70	**70**	80	**80**	**150**	≠ 600
2	70	**140**	80	**160**	**300**	≠ 600

Now we make a guess and check table. Students try 2 hours next; which brings the total distance of the two cars to 300, still short of 600. No

Moving right along…

Time	Car A Rate	Car A Distance	Car B Rate	Car B Distance	Total	= 600?
1	70	**70**	80	**80**	**150**	≠ 600
2	70	**140**	80	**160**	**300**	≠ 600
3	70	**210**	80	**240**	**450**	≠ 600

At three hours, still short

Success at last!

Time	Car A Rate	Car A Distance	Car B Rate	Car B Distance	Total	= 600?
1	70	**70**	80	**80**	**150**	≠ 600
2	70	**140**	80	**160**	**300**	≠ 600
3	70	**210**	80	**240**	**450**	≠ 600
4	70	**280**	80	**320**	**600**	= 600

At 4 hours, we have success.

Now, Guess and check is an arithmetic approach and it certainly would not be useful for problems that have decimal solutions like 4.56. But it can be used as a launch pad to get them from arithmetic thinking to algebraic thinking. For one thing, in setting up the table, the concept that the sum of the distances is equal to 600 is reinforce and for another, the concept that the time for both cars will be the same is reinforced. In that vein, we ask them how far will Cars A and B travel over t hours

Putting it all together with algebra

Time	Car A Rate	Car A Distance	Car B Rate	Car B Distance	Total	= 600?
1	70	**70**	80	**80**	**150**	≠ 600
2	70	**140**	80	**160**	**300**	≠ 600
3	70	**210**	80	**240**	**450**	≠ 600
4	70	**280**	80	**320**	**600**	= 600
t	70	?	80	?	?	

In that vein, we ask them how far will Cars A and B travel over t hours

They should know this by now given that they had it in earlier problems, so they get their retrieval practice. But with the new info they're processing some students may need reminding

Putting it all together with algebra

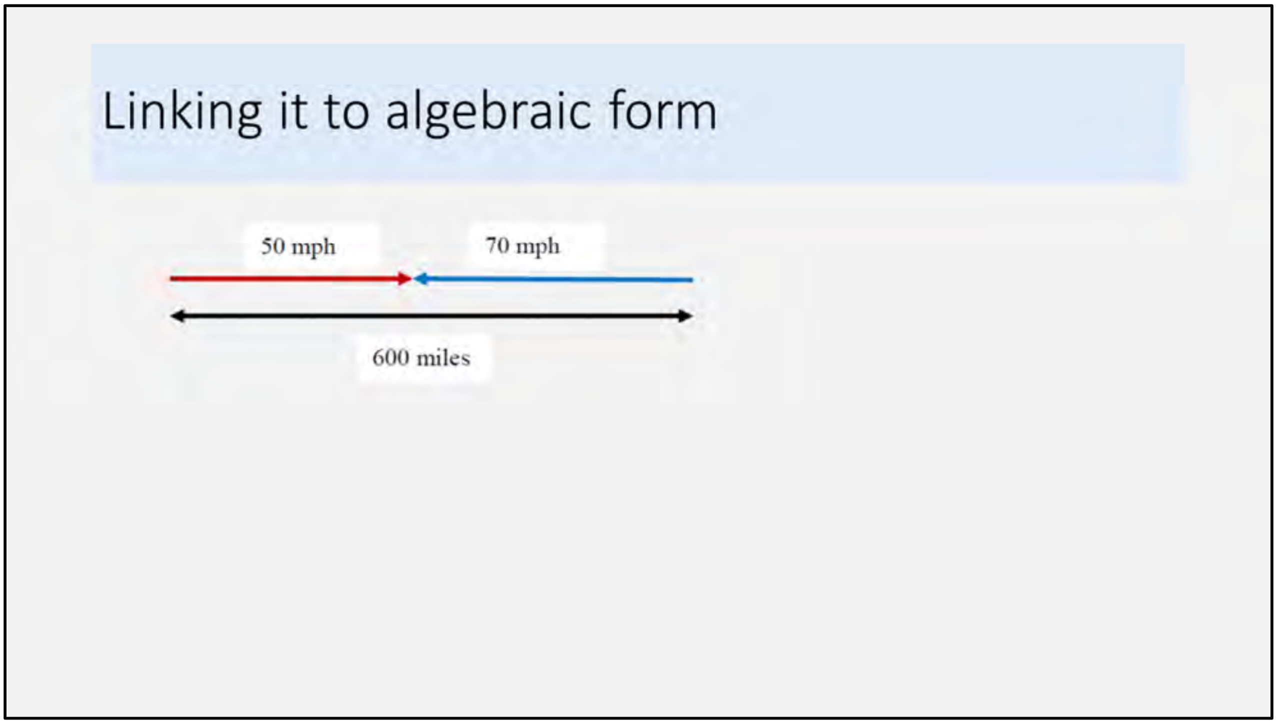
Time	Car A Rate	Car A Distance	Car B Rate	Car B Distance	Total	= 600?
1	70	**70**	80	**80**	**150**	$\neq 600$
2	70	**140**	80	**160**	**300**	$\neq 600$
3	70	**210**	80	**240**	**450**	$\neq 600$
4	70	**280**	80	**320**	**600**	$= 600$
t	70	$70t$	80	$80t$	**600**	

$$70t + 80t = 600 \rightarrow t = 4 \text{ hrs}$$

And they fill out the last line.

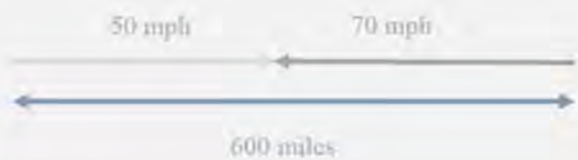
And now they repeat what they did before, but this time add the two distance letting algebra do the heavy lifting, which gives them an equation for them to solve.

When I first started teaching algebra I thought they would immediately see how algebra is a much quicker way. But it takes some effort to move away from arithmetic thinking.



But then we cut out all guesswork. We give them a similar problem

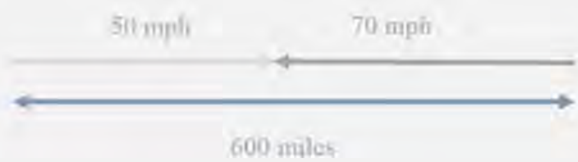
Linking it to algebraic form



	Rate	×	Time	=	Distance
Car A	50		t		?
Car B	70		?		?
			Total Distance:		?

But instead of a guess and check table, students use this table. This table will now show up in subsequent problems. It organizes “what they know” about the problem and “what they don’t know” in terms of a variable. We give them some info and have them fill in the rest.

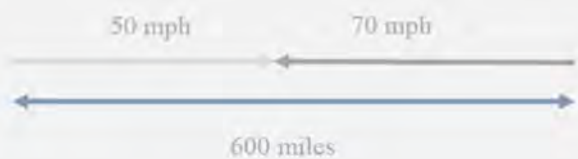
Linking it to algebraic form



	Rate ×	Time =	Distance
Car A	50	t	**50t**
Car B	70	**t**	**70t**
		Total Distance:	**600**

Which they do. The distance is now expressed as rate x time: that is, 50t and 70t. The information needed to solve the problem is there, and it is now up to the student to know "what to do with what"

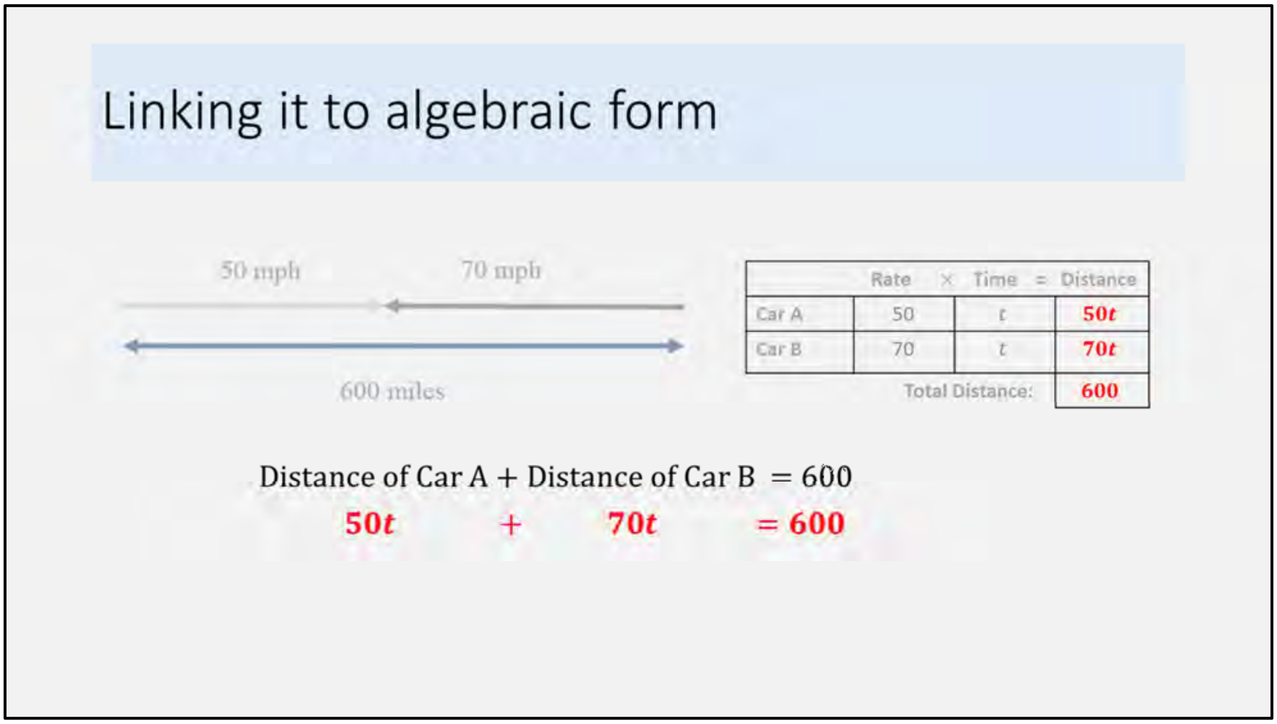
Linking it to algebraic form



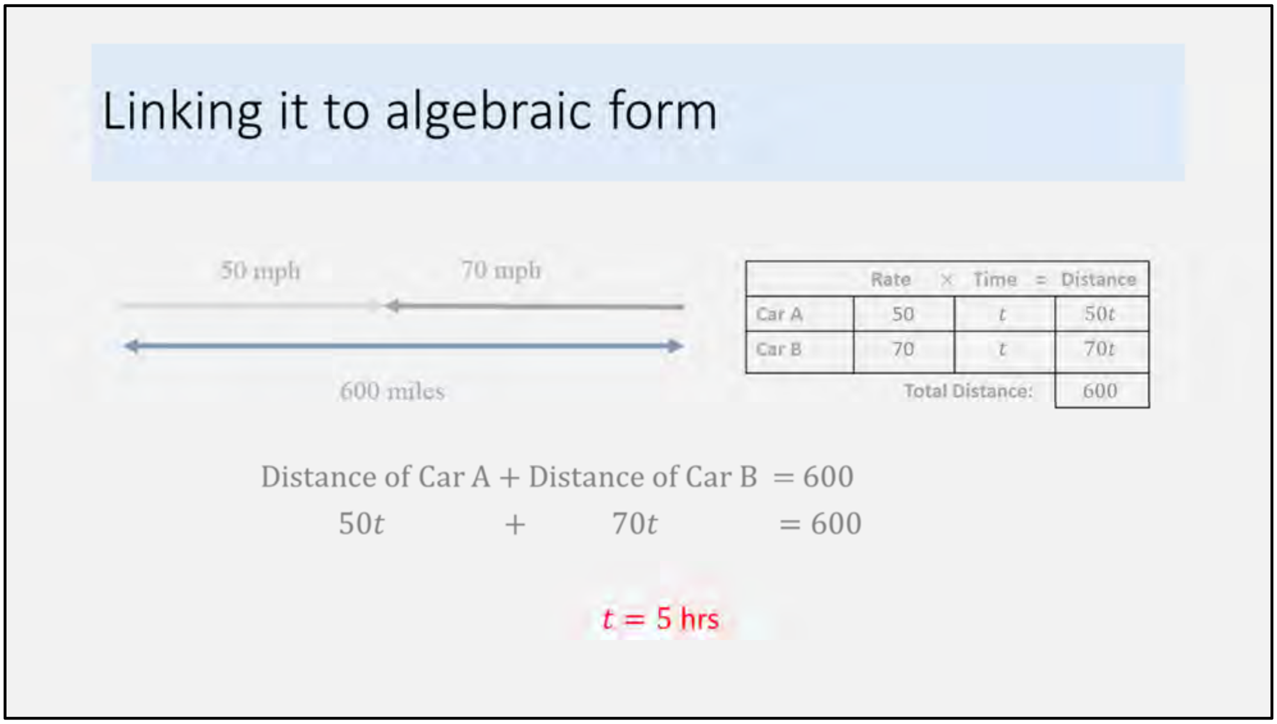
	Rate ×	Time =	Distance
Car A	50	t	$50t$
Car B	70	t	$70t$
		Total Distance:	600

$$\text{Distance of Car A} + \text{Distance of Car B} = 600$$

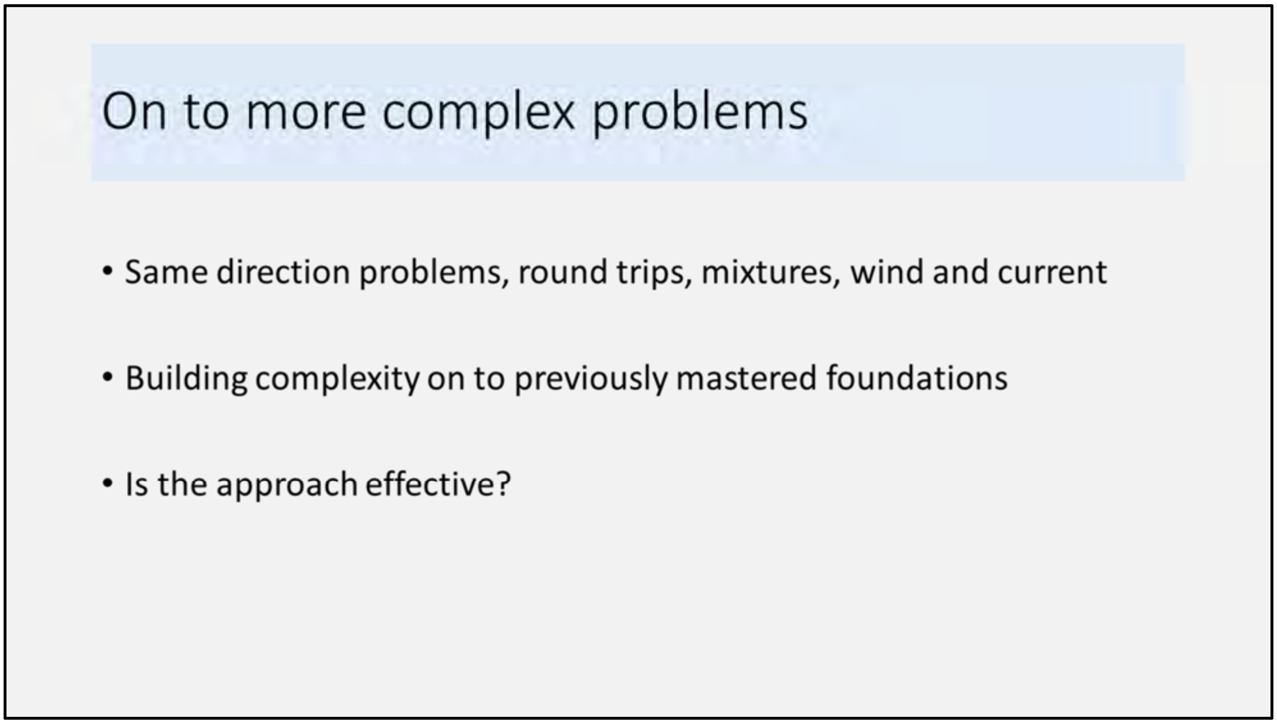
They know from the guess and check method they did previously that the sum of the distances of the two cars = total distance, or 600 miles in this case.



So they now connect it to the information in the table, and translate it into an equation: $50t + 70t = 600$



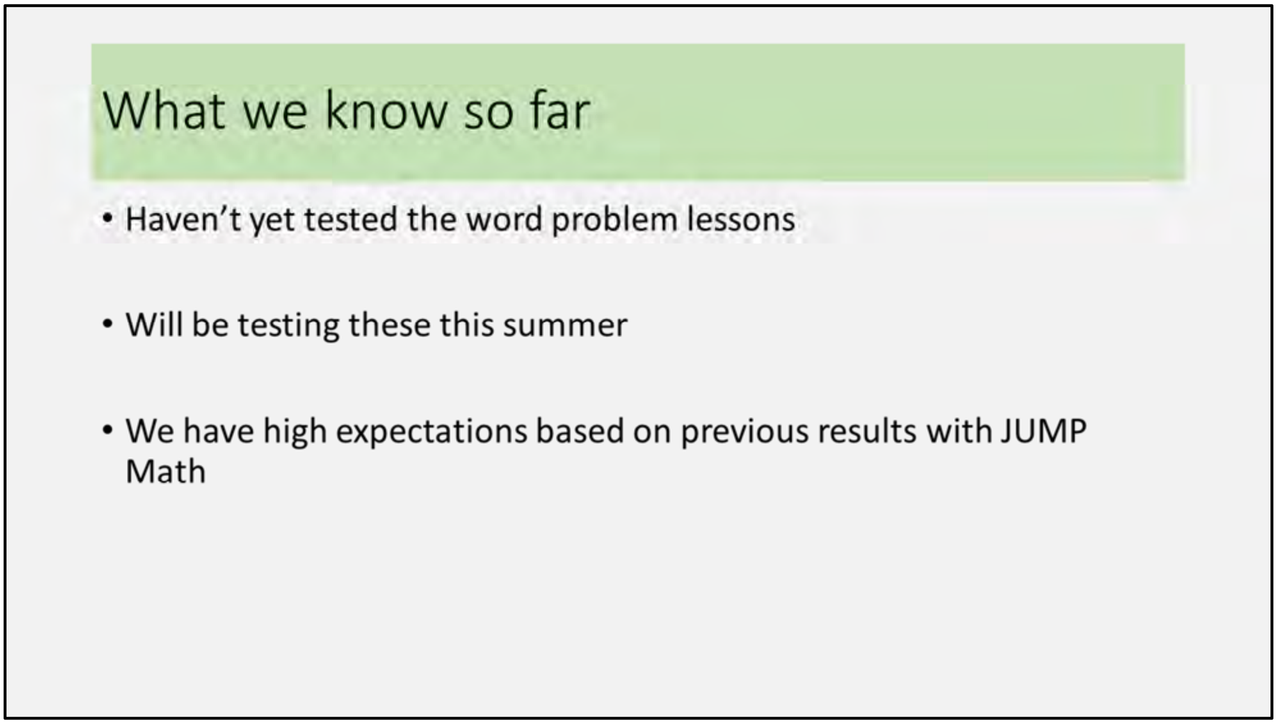
… and solve. I'll tell them that this problem is the same as the "moving away" problem but in reverse, which now they begin to see. Students will then solve similar problems, both moving away and toward, time to meet given, but not speeds of objects – all using the table we introduced to give them practice with it. If we left them to their own devices, many would go back to guess and check, since arithmetic methods are likely to be their preferred non-conscious choice.



More situations follow: catch-up problems, round trips, mixture problems, wind and current.

Building complex variations of previous mastered foundations.

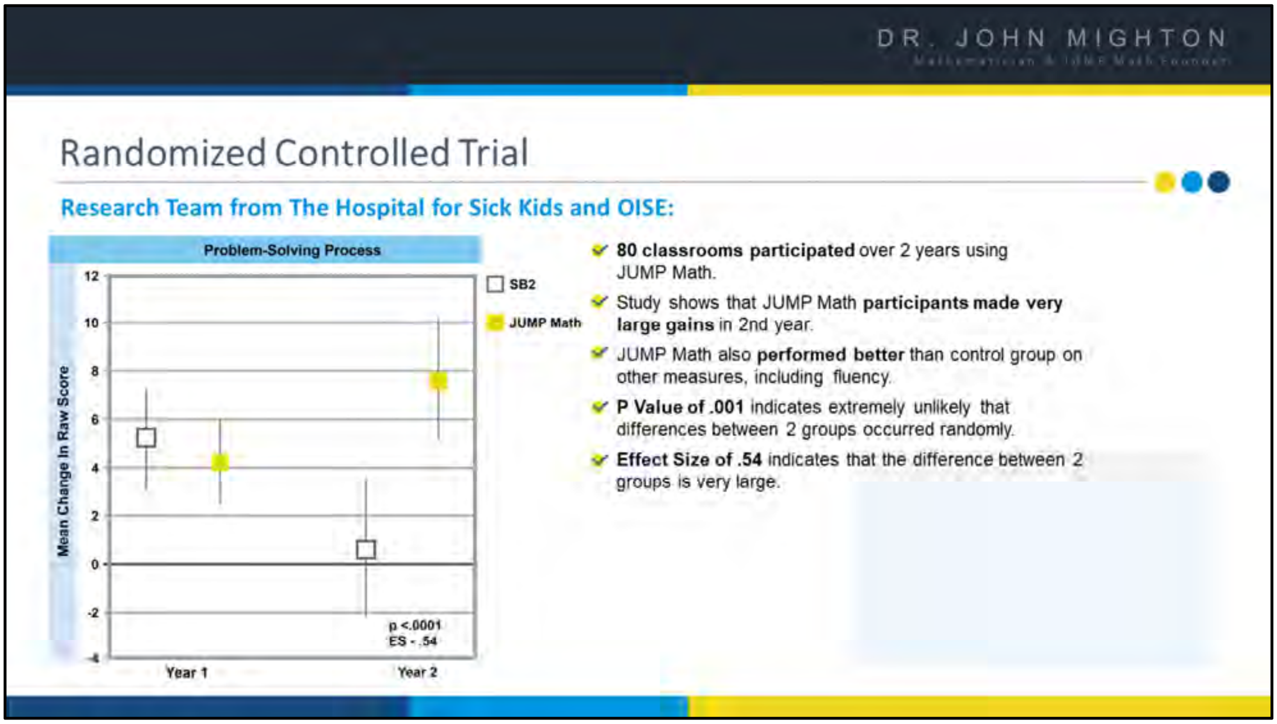
So the question you may be asking yourselves is whether the approach we've described is effective, and if so, how effective. John will now talk about our expectations for the algebra program.



Unfortunately we have not yet tested the word problem lessons in time for this conference.

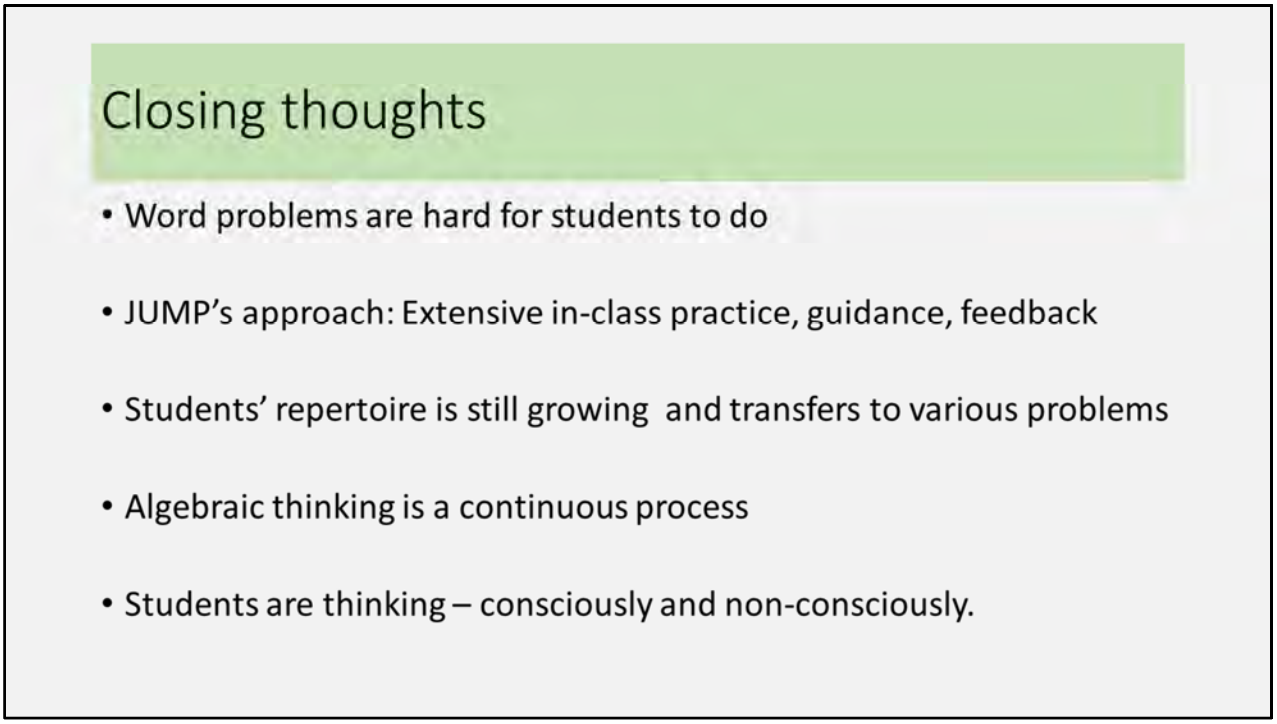
We will be undertaking some testing of the word problem lessons this summer on college students in need of remediation, as well as high school students taking algebra for the first time.

But we do have some expectations based on previous results with JUMP Math



We know for example that for our K-8 program, we have had excellent results as shown by this exhibit. The variation and scaffolding, and explicit instruction we use in JUMP can build students' mathematical thinking and proficiency as evidenced by the results of these trials.

We have seen consistent improvement over a five year period. We expect that what is working in our K-8 programs will have similar results in algebra.



And now for some closing thoughts that summarize what we've talked about, plus some reflections. Word problems are admittedly hard for students to do. A an algebra teacher with more than 30 years experience told me "Most students don't get really good at algebraic word problems in their first algebra course."

We believe that our approach will get better results than methods provided in current textbooks which have cursory explanations, few worked examples, and few practice problems; and those that they have are either too easy or ramp up in difficulty very quickly.

JUMP's problem sequence is designed to build students' repertoire and allows students to transfer their knowledge to problems that use similar problem structures.

Learning to solve problems is a continuous process, and so is the acquisition of algebraic thinking. It isn't done overnight; it takes practice, experience, and time. In the meantime, students are not just imitating algebraic schema without knowing what they're doing. They are making choices with what they have learned – consciously and non-consciously, and thinking mathematically with them.

Questions

- Feel free to contact us:

- Barry Garelick: barryg99@yahoo.com
- John Mighton: jm@jumpmath.org

"Remember how I've always had a hard time asking for help?"

We're happy to answer any questions you may have. And feel free to contact via email at these email addresses.